Post-Newtonian Equilibrium Solutions to the Three-Body Problem in General Relativity

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by

Kei Yamada

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Abstract

Recently, many projects of the second generation gravitational wave detectors including Advanced LIGO (USA), Advanced VIRGO (France and Italy), and KAGRA (Japan) and of the high-precision astrometric observations such as Gaia (Europe) and JASMINE (Japan) are planned. By these projects, it is expected to become possible to test general relativity through the strong gravitational field and N-body systems. For testing general relativity, it is important to investigate the general relativistic effects on the dynamics of the bodies. As analytical equilibrium solutions to the three-body problem in Newtonian gravity, Euler's collinear solution and Lagrange's equilateral triangular one exist. We analytically study the post-Newtonian (PN) effects on these solutions for general masses in circular motion in the framework of general relativity. First, we show that a PN collinear configuration corresponding to Euler's one remains an equilibrium solution with the general relativistic corrections to the distances between the bodies in the first post-Newtonian (1PN) approximation. Also, we prove the uniqueness of the PN collinear configuration for given system parameters. This means that the number of the collinear Lagrangian points is not changed from the Newtonian case. Next, we show that an equilateral triangular solution exists at the 1PN order in only two cases: (i) three finite masses are equal and (ii) one mass is finite and the others are zero. We also consider a PN triangular solution for general masses with the general relativistic corrections to the distances between the bodies. In addition, we investigate a linear stability of the PN triangular solution and derive the condition for stability at the 1PN order. Furthermore, we estimate the magnitude of the PN corrections in the solutions and show that the common angular velocity of the bodies is smaller than in the Newtonian case for both the solutions. These results are useful not only to test general relativity through the high-accurate astrometry observations but also to study gravitational waves emitted from the general relativistic three-body systems.

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Notation

The speed of light c and the Newtonian gravitational constant G are always written explicitly. The solar mass is denoted by $M_\odot$.

Greek indices, such as $\alpha, \beta, \cdots$ or $\mu, \nu, \cdots$, take the values $0, \cdots, 3$, while spatial indices are denoted by Latin letters, $i, j, \cdots = 1, 2, 3$, unless otherwise stated. We give a number to particles or bodies by Latin capital letters, such as $A, B, C, \cdots$ or $I, J, K, \cdots$. The flat space metric is

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1).$$

We also define

$$\begin{aligned} x^\mu &= (x^0, \boldsymbol{x}), \quad x^0 = ct, \\ d^4x &= dx^0 d^3x = c\,dt\,d^3x, \\ u^\mu &= \frac{dx^\mu}{d\tau}, \end{aligned}$$

where τ is a proper time. And we employ the Einstein summation convention, such as

$$\eta_{\mu\nu} x^\mu = \sum_{\mu=0}^{3} \eta_{\mu\nu} x^\mu = \eta_{0\nu} x^0 + \eta_{1\nu} x^1 + \eta_{2\nu} x^2 + \eta_{3\nu} x^3. \tag{1}$$

We denote order-of-magnitude estimations and approximate values by $\sim$ and $\approx$, respectively, for instance,

$$M_\odot \sim 10^{30}\ [\mathrm{kg}], \quad M_\odot \approx 1.988 \times 10^{30}\ [\mathrm{kg}].$$

Also, we express approximate equations by $\simeq$, such as

$$e^\epsilon \simeq 1 + \epsilon, \quad \text{for } |\epsilon| \ll 1.$$

Chapter 1

Introduction

1.1 Testing general relativity

One of the most promising tests of gravitational theory in the near future is a direct detection of gravitational waves (GWs). There is an analogy between the generation of GWs and that of electromagnetic waves. However, because of weakness of gravitation, GWs have not been directly detected, while they were indirectly detected. Indeed, the observation of the orbital decay in the binary pulsar PSR B1913+16 is in agreement with the theoretical prediction taking account of the gravitational radiation reaction in general relativity [12, 81, 84, 56, 126]. Currently, to detect GWs from general relativistic objects and the early universe, the second generation of ground-based detectors are planned around the world: in the USA (Advanced LIGO [6]), in Italy (Advanced VIRGO [13]), and in Japan (KAGRA [72]), for instance. In addition, there are plans of the future space-borne detectors, such as DECIGO in Japan and eLISA in Europe [37, 45]. Within a few years, a new window to test gravitational theory will open with the first direct detection of GWs by these detectors.

In order to extract astrophysical information from GWs, an accurate data analysis strategy like matched filtering is required. For such a style of data analysis, we must make templates of GW signals. Namely, it is very important to predict the gravitational waveforms, and hence, we need to understand the dynamics of the GW sources. GWs are generated by dynamical astrophysical events. When compact stars such as neutron stars (NSs) or black holes (BHs) are involved in such events and they are sufficiently near to us, GWs are expected to be strong enough to be detected. In particular, coalescing binaries of compact objects, such as a NS-NS binary, are the most promising sources of GWs for the grand-based detectors. Indeed, statistical studies have predicted that the detection rate of GWs from these will be 40 events per year [1], and the waveforms can be predicted with a high accuracy [17, 102, 126].

As another point of view of testing gravity, high-precision astrometry observations are useful. Recently, space astrometry mission are planned around the world: the Gaia mission in Europe [52] and the JASMINE mission in Japan [71], for instance. Gaia has been planned as a successor to Hipparcos mission, and it will perform all-sky surveys in optical bands. On the other hand, complementarily to Gaia, the JASMINE mission will survey the bulge of the Milky Way, which cannot be reached by Gaia, in infrared bands. It is expected that there are supermassive BHs in galactic centers and many-body systems are composed around there (e.g. Ref. [7]). Also, JASMINE will observe specific objects, such as Cygnus X-1, a candidate of a BH. These missions are expected to derive reliable distances and tangential velocities of stars located within 10 kiloparsec (kpc) distant from us with the accuracy of 10 microarcseconds (μas).

If a star is part of a binary system or has a planetary system, it is important to derive information of their orbital motion. JASMINE will observe the same star every day, while Gaia will observe it once a month. Hence, the time resolution of JASMINE is more better than that of Gaia, and this is advantageous to derive information of dynamics of binaries. The key of orbit determinations by analytical methods for binaries is finding the position of the common center of mass. For the solar planets and comets, we can safely assume that the common center of mass is the Sun, and then the orbit can be immediately determined. For visual binaries, formulations for orbit determinations have been well developed since the nineteenth century [117, 15, 2, 36, 99]. In the twenty-first century, Asada and his collaborators found an analytical formula of orbit determinations for astrometric binaries, where one object is unseen [10, 11, 8]. Yamada et al. [133] improved a moment approach proposed by Iwama, Asada, and Yamada [63] for astrometric binaries with a low signal-to-noise (SN) ratio. At present, numerical methods are also successfully used [43, 93, 27].

In this way, from the two points of view as *GW astronomy* and *high-precision astrometry*, highly precise observational data, which are helpful to test gravitational theories and to seek the true theory of gravity, will be available. However, for three- (or more-) body systems, investigation of the dynamics of the bodies has not been developed even in general relativity. Therefore, it is very important to understand the dynamics of the bodies in various modified gravity as well as in general relativity.

1.2 Theoretical methods

The orbital evolution, the mergers, and the tidal disruption of close binaries are strongly affected by the general relativistic effects. In such a regime, where it is difficult to understand the dynamics by analytical methods, the *numerical relativity* is required. For binary BHs, the first stable simulation was performed in 2005 [95]. The mergers of BH-NS binaries were also studied (e.g. [108, 109, 107, 46, 42]). The first fully general relativistic simulation for BH-NS binary mergers was performed by Shibata et al. [108], extending their earlier works for NS-NS binaries (e.g. [106]). Other groups also performed simulations for BH-NS binary mergers, by extending earlier works on NS-NS binaries and BH-BH binaries (e.g. [41, 22]). By the numerical simulations, the effects of the equation of state of NS and spins of BH and NS on the orbital evolution and gravitational waveforms were accurately predicted. However, the numerical relativity is still developing and the parameter space is quite large. For instance, binaries with large BH spin have not been studied yet. And also, there is a few study taking account of the inclination of spin. For this, analytical study is of importance to understand the dynamics of bodies and to predict gravitational waveforms, complementarily to numerical methods.

As an analytical approach which is valid in strong gravitational fields, the BH perturbation approach is used. In this approach, we consider a particle of mass μ orbiting a BH of mass M assuming $\mu \ll M$. Hence, this approach is valid in the case of binaries with the small mass ratio μ/M. The BH perturbation approach takes full account of general relativistic effects of the background spacetime and arbitrary orbits of a body of small mass. Therefore, we could understand how and when general relativistic effects become important by comparing with numerical simulations. In addition, the BH perturbation approach is applicable to GW events of compact object orbiting supermassive BHs in galactic centers. Actually, such a event is one of the main targets of eLISA. Because of the applicability of the BH perturbation approach to general relativistic orbits, the location of the inner-most stable circular orbit (ISCO) is important. In the Schwarzschild BH case, this is given by $r_{\rm ISCO} = 6GM/c^2$. Marginal stable circular orbits (MSCOs), such as the ISCO, in any spherically symmetric and static spacetime is studied by Ono et al. [94].

In the BH perturbation approach, for the Schwarzschild BH, a single master equation for the metric perturbations for the odd parity was derived by Regge and Wheeler in 1957 [97]. Thirteen years later, Zerilli found the one for the even parity [134]. These equations (Regge-Wheeler-Zerilli equations) reduce to the Klein-Gordon equations in the limit of the flat space-time. Master equations for any spherical space-time in the Horndeski theory, which is the most general scalar-tensor theory with second-order field equations, were derived by Kobayashi and his collaborators

[74, 75]. However, such equations for the Kerr BH have not been found so far.

For the first time, a master equation for the curvature perturbations was derived by Bardeen and Press [14] for the Schwarzschild BH, and by Teukolsky [116] for the Kerr BH. The master equations (Teukolsky equations) do not reduce to the Klein-Gordon equations even in the flat-spacetime limit. Chandrasekhar showed that the Teukolsky equations can be transformed to the Regge-Wheeler-Zerilli equations in the Schwarzschild BH case [23]. Sasaki and Nakamura generalized this for the Kerr BH case [100, 101]. Solving the master equations, we can investigate a particle motion in a strong gravitational field and GWs emitted during its orbital evolution (for a particle in quasi circular motion around the Schwarzschild BH, see Ref. [114, 115], for instance).

Another analytical method is the post-Newtonian (PN) approach. For a system where orbital separations between the bodies are sufficiently large compared with their expanses, we may neglect the expanses and internal structure of the bodies, regarding them as point-like particles. In such a case, the dynamics of the system can be analytically understood by the PN approach. At the first post-Newtonian (1PN) order, the first general relativistic terms were derived by Lorentz and Droste [80]. The 1PN corrections to the equations of motion, which can be applicable to any compact objects (e.g. NS, BH, and naked singularity), were obtained for N-body systems by Einstein, Infeld, and Hoffmann [44]. Ohta and his collaborators discussed a Hamiltonian for N-body systems at the 2PN order [88, 90, 89]. At the 2.5PN order, Damour and Deruelle derived equations of motion in the harmonic coordinates [32, 31, 28, 29]. The energy and the angular momentum of systems to terms order of the 2PN are conservative. Non-conservative effects by gravitational radiations appear at the 2.5PN order for the first time.

In the PN approximation, bodies are usually assumed to be point-like particles. The 2.5PN equations of motion for binaries in the harmonic coordinates where the bodies have strong internal gravity were derived by Itoh, Futamase, and Asada [61, 62]. They showed by developing earlier work by Futamase and Schutz [50, 51, 49] that the equations are in complete agreement with the Damour-Deruelle equations [32, 31] for point-like particles. The equations of motion to the order of 2.5PN [32, 31, 28, 29] have been used to investigate the radiation damping of the binary pulsar [29, 30, 35]. For the 3PN equations of motion, several groups using different methods, for instance, the Arnowitt-Deser-Misner (ADM) Hamiltonian canonical formalism of general relativity [66, 67, 68], the PN iteration [18, 20, 19, 21], and the surface-integral method [60, 59], reported equivalent results. The 3.5 PN terms in the equations of motion for binaries were derived by Iyer and Will [64, 65]. The equations of motion for point-particle binaries in the 4PN approximation have been reported [48, 69, 70, 34].

The PN formalism can be extended to various metric theories as well as general relativity.

The extension is known as the parameterized post-Newtonian (PPN) formalism and the parameters are called the PPN parameters [126, 125, 84]. The PPN formalism was initiated by Nordvedt, who studied the PN metric for point-like particles by extending the earlier work by Eddington, Robertson and Schiff [87, 125]. Will generalized this for a perfect fluid [124]. The general unified PPN formalism was derived by Will and Nordtvedt [127]. Alexander and Yunes showed that a new PPN parameter appears in the Chern-Simons (CS) gravity [4, 3, 5]. The CS parameter has been restricted by satellite experiments, such as Gravity Probe B and LAGEOS [4, 3, 112]. Okawara, Yamada, and Asada studied the constraint on the CS parameter from terrestrial experiments using interference effects of matter waves [91, 92]. Also, Kikuchi et al. investigated the relativistic Sagnac effects by the CS gravity [73].

1.3 Importance of the three-body problem

As mentioned above, binary systems have been enthusiastically studied by using both analytical and numerical methods. In fact, general relativity has been tested with a very high accuracy in the solar system and the binary pulsar PSR B1913+16, for instance. As a new test of general relativity, it is interesting to take account of general relativistic three- (or many-) body interactions. Actually, the PN terms associated with many-body interactions appear in the equations of motion at the 1PN order. The terms cannot be considered in the relativistic two-body problem nor the Newtonian many-body problem.

The three-body problem is not integrable by analytical methods even in Newtonian gravity. As particular solutions, however, Euler and Lagrange found a collinear solution and an equilateral triangular one, respectively [47, 77]. These are *equilibrium solutions*, where the centrifugal force balances with the gravitational one for each body. The solutions to the restricted three-body problem, where one of the three bodies is considered a (massless) test particle, are known as the Lagrangian points $L_1, \cdots, L_5$ [36, 54]. The Solar and Heliospheric Observatory (SOHO) launched by ESA/NASA and Planck launched by ESA are in operation at the Sun-Earth L_1 and L_2, respectively. Lagrange's equilateral triangular solution has also a practical importance since L_4 and L_5 for the Sun-Jupiter system are stable points and indeed the Trojan asteroids are located there. For the Sun-Earth system, asteroids were also found around L_4 by recent observations [26]. Numerical solutions, such as a figure-eight and Hénon's criss-cross orbits, are also found for three equal masses [85, 24, 55, 113].

Recently, the Lagrangian points have attracted renewed interests in relativistic astrophysics [76, 82, 105, 103, 9, 119], where they have discussed PN corrections to the Lagrangian points

[76, 82] and the gravitational radiation reaction on the particle at L_4 and L_5 analytically [9] and by numerical methods [105, 103, 119]. It is currently important to reexamine the Lagrangian points in the framework of general relativity. As a pioneering work [86], Nordtvedt has pointed out that the locations of the triangular points are very sensitive to the ratio of the gravitational mass to the inertial one though his analysis does not fully take account of the 1PN terms. Along this course, it might be important as a gravity experiment to discuss the three-body coupling terms in the PN force because some of the PN terms are proportional to a product of three masses such as $m_1 \times m_2 \times m_3$. In addition, it has been pointed out that three-body interactions might play important roles in compact binary mergers in hierarchical triple systems [16, 83, 123, 118, 104]. Very recently, a first relativistic hierarchical triple system has been discovered by Ransom et al. [96]. In addition, for such a relativistic hierarchical triple system, PN effects on the perihelion shift of the outer third body were investigated by Yamada and Asada [131].

For three finite masses, in the 1PN approximation, the existence and uniqueness of a PN collinear solution corresponding to Euler's one have been shown by Yamada and Asada [129, 130]. Also, Ichita, Yamada and Asada have shown that an equilateral triangular solution is possible at the 1PN order if and only if all the three masses are equal [57]. Generalizing this earlier work, Yamada and Asada have found a PN triangular equilibrium solution for general masses with 1PN corrections to each side length [132]. This PN triangular configuration for general masses is not always equilateral and it recovers the previous results [76, 82] for the restricted three-body problem. For a figure-eight solution, Imai, Chiba, and Asada studied the PN effects on the orbit and the gravitational waveform [58, 25].

In Newtonian gravity, Gascheau proved that Lagrange's equilateral triangular configuration for circular motion is stable in some cases of mass ratio [53]. Routh extended the result to a general law of gravitation $\propto 1/r^k$ [98]. For the restricted three-body systems, the stability of L_4 (L_5) has been studied in the 1PN approximation [40, 110, 111]. These results imply that a triple system with a triangular configuration may be a candidate of GW source.

1.4 The aims of this study

In this dissertation, following the Refs. [129, 130, 57, 132], we study PN equilibrium solutions corresponding to Euler's and Lagrange's ones to the three-body problem in general relativity. And also, the properties of the PN effects on these solutions are discussed by comparing with the Newtonian cases.

This dissertation is organized as follows. In Chapter 2, we summarize the Newtonian two-body problem and Euler's and Lagrange's solutions to the Newtonian three-body problem. In Chapter 3, we derive the 1PN equations of motion, called the Einstein-Infeld-Hoffmann equations of motion, for N-body systems. Chapter 4 is devoted to a derivation of the PN collinear solution corresponding to Euler's one. In Chapter 5, we consider the PN triangular solution, which is the general relativistic version of Lagrange's solution. We conclude in Chapter 6. In Appendix A, we show that the expression for the PN center of mass reduces to the Newtonian one in the circular equilateral triangle even at the 1PN order. Appendix B is devoted to the PN corrections to the common angular velocity of the collinear configuration. In Appendix C, we consider the condition for stability of the PN triangular solution by eigenvalue analysis in a collaboration with Tsuchiya and Asada. Appendix D is a preliminary note of gravitational radiation reactions on Lagrange's solution in a collaboration with Asada, Iseki, and Harada.

Chapter 2

Celestial Mechanics in Newtonian Gravity

In this chapter, we discuss a system consisting of two bodies in Newtonian gravity and summarize Euler's and Lagrange's solutions to the Newtonian three-body problem. For the Newtonian two-body problem, the complete analytical solution of the equations of motion have been obtained. This solution can be divided into the three trajectory cases, elliptic, parabolic, and hyperbolic orbits. We concentrate on the elliptic case. For the three-body problem, we focus on circular motion. The treatment of the problem in this chapter follows Refs. [36, 54, 79, 121].

2.1 The two-body problem in Newton gravity

2.1.1 The complete solution

First, let us consider how the problem can be simplified by separating the motion of the system into the motion of the center of mass and that of the bodies relative to the center of mass.

We assume that the potential energy of two bodies interacting each other depends only on the distance between the bodies, that is, on the magnitude of the difference between the position vectors. The Newtonian gravitational potential satisfies the assumption. Then, the Lagrangian of the system with the potential $U(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)$ is expressed as

$$\mathscr{L} = \frac{m_1 \dot{\boldsymbol{r}}_1^2}{2} + \frac{m_2 \dot{\boldsymbol{r}}_2^2}{2} - U(|\boldsymbol{r}_1 - \boldsymbol{r}_2|), \tag{2.1}$$

where m_I and $\boldsymbol{r}_I$ are the mass and the position vector of the Ith body, respectively, and the dot denotes the differentiation with respect to time t.

We express the relative position vector of the bodies as

$$\boldsymbol{r} \equiv \boldsymbol{r}_1 - \boldsymbol{r}_2 \tag{2.2}$$

and choose the position of the center of mass as the origin of the coordinates:

$$m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 = \mathbf{0}. \tag{2.3}$$

Then, we obtain

$$\boldsymbol{r}_1 = \frac{m_2}{m_1+m_2}\boldsymbol{r}, \tag{2.4}$$

$$\boldsymbol{r}_2 = -\frac{m_1}{m_1+m_2}\boldsymbol{r}. \tag{2.5}$$

Substituting these expressions into (2.1), we find

$$\mathscr{L} = \frac{\mu\dot{\boldsymbol{r}}^2}{2} - U(r), \tag{2.6}$$

where $r = |\boldsymbol{r}|$ and

$$\mu = \frac{m_1 m_2}{m_1+m_2} \tag{2.7}$$

is the reduced mass. The function (2.6) is formally identical with the Lagrangian of a single body with mass μ moving in the spherically symmetric gravitational field.

Next, we consider the motion of the body with the Lagrangian (2.6). The angular momentum of the system relative to the center of the field is conserved. The angular momentum for the body is expressed as

$$\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{p}, \tag{2.8}$$

where $\boldsymbol{p}$ denotes the momentum of the body. Since the vector $\boldsymbol{J}$ is perpendicular to $\boldsymbol{r}$, the fact that $\boldsymbol{J}$ is constant shows that during the motion of the body, its position vector always remains in the plane perpendicular to $\boldsymbol{J}$. Thus, the path of the body lies completely in the plane. Using polar coordinates (r,φ) in the plane, the Lagrangian is expressed as

$$\mathscr{L} = \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r). \tag{2.9}$$

This function does not involve the coordinate φ explicitly. Then, the Euler-Lagrange equation for $\dot{\varphi}$ becomes

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{\varphi}} = \frac{\partial \mathscr{L}}{\partial \varphi} = 0. \tag{2.10}$$

Thus, we can check the conservation law of angular momentum:

$$J = \mu r^2 \dot{\varphi} = \text{constant}. \tag{2.11}$$

The complete solution to the problem of the motion of the body can be obtained by using the conservation laws of the energy E and the angular momentum J without a direct use of the equations of motion. Substituting the expression of $\dot{\varphi}$ of J from Eq. (2.11) into the expression of the energy integral

$$E = \dot{r}\frac{\partial \mathscr{L}}{\partial \dot{r}} + \dot{\varphi}\frac{\partial \mathscr{L}}{\partial \dot{\varphi}} - \mathscr{L}, \tag{2.12}$$

we obtain

$$\begin{aligned} E &= \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) + U(r) \\ &= \frac{\mu \dot{r}^2}{2} + \frac{J^2}{2\mu r^2} + U(r). \end{aligned} \tag{2.13}$$

Therefore,

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{\mu}[E - U(r)] - \frac{J^2}{\mu^2 r^2}} \tag{2.14}$$

or

$$t = \int \frac{dr}{\sqrt{\frac{2}{\mu}[E - U(r)] - \frac{J^2}{\mu^2 r^2}}} + \text{constant}. \tag{2.15}$$

Furthermore, Eq. (2.11) can be rewritten as

$$d\varphi = \frac{J}{\mu r^2} dt. \tag{2.16}$$

Substituting the expression into Eq. (2.14), it can be integrated and we obtain

$$\varphi = \int \frac{(J/r^2)dr}{\sqrt{2\mu[E - U(r)] - \frac{J^2}{r^2}}} + \text{constant}. \tag{2.17}$$

Equations (2.15) and (2.17) are the general solution to the problem.

2.1.2 The Kepler problem

Since the Newtonian gravity is an attractive force, the gravitational potential is expressed as

$$U = -\frac{\alpha}{r}, \tag{2.18}$$

where $\alpha = Gm_1m_2$. The orbit equation can be obtained from the general formula (2.17). Substituting Eq. (2.18) and integrating, we obtain

$$\varphi = \cos^{-1} \frac{\frac{J}{r} - \frac{\mu\alpha}{J}}{\sqrt{2\mu E + \frac{\mu^2\alpha^2}{J^2}}} + \text{constant}. \tag{2.19}$$

We choose the origin of φ such that the constant is zero, and we put

$$p = \frac{J^2}{\mu\alpha}, \tag{2.20}$$

$$e = \sqrt{1 + \frac{2EJ^2}{\mu\alpha^2}}. \tag{2.21}$$

Then the orbit equation can be written as

$$\frac{p}{r} = 1 + e\cos\varphi. \tag{2.22}$$

This is an equation of a conic section with one focus at the origin of the coordinates. p and e are called the *latus rectum* and the *eccentricity* of the orbit, respectively. Equation (2.22) shows that r becomes the *pericenter* when $\varphi = 0$ in our choice of the origin of φ.

If the eccentricity $e < 1$ (i.e. $E < 0$), then the orbit is an ellipse and the motion is finite and called the *Kepler motion*. And if the eccentricity $e = 0$ where the energy is the minimum value, the ellipse becomes a circle. Figure 2.1 shows an elliptic orbit. In this figure, a_e and b_e denote the semi-major and semi-minor axes, respectively, and the blue disks are the focuses. The semi-major and semi-minor axes are expressed as

$$a_e = \frac{p}{1-e^2} = \frac{\alpha}{2|E|}, \quad b_e = \frac{p}{\sqrt{1-e^2}} = \frac{J}{\sqrt{2\mu|E|}}, \tag{2.23}$$

respectively. And also, the pericenter distance r_p and the apocenter distance r_a are

$$r_p = \frac{p}{1+e} = a_e(1-e), \quad r_a = \frac{p}{1-e} = a_e(1+e), \tag{2.24}$$

respectively. If $E \geqq 0$, the motion is infinite. If $E = 0$, then the eccentricity $e = 1$ and the body moves along a parabola. And, if $E > 0$, then the eccentricity $e > 1$ and the orbit becomes a hyperbola.

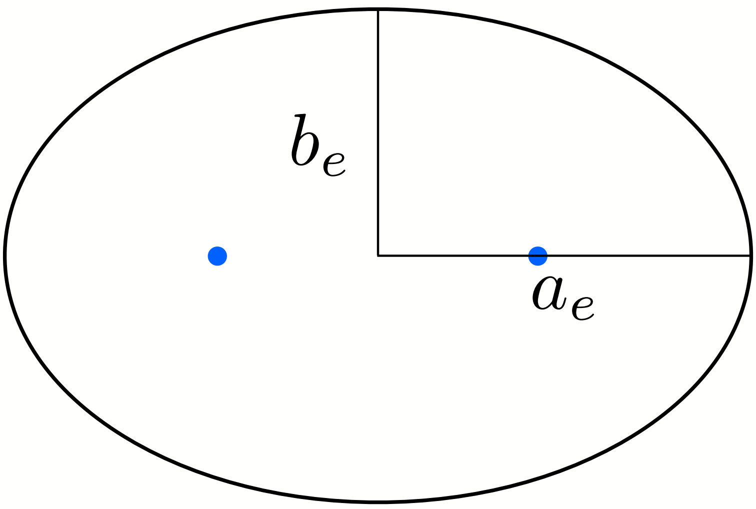

Figure 2.1: An elliptic orbit.

2.1.3 The initial-value problem in elliptic motion

Let us study how the initial velocity are determined in the Kepler motion. The equations of motion for two bodies in the Cartesian coordinates (x, y) are

$$m_1\frac{d^2x_1}{dt^2} = -\frac{Gm_1m_2(x_1-x_2)}{[(x_1-x_2)^2+(y_1-y_2)^2]^{3/2}}, \quad m_1\frac{d^2y_1}{dt^2} = -\frac{Gm_1m_2(y_1-y_2)}{[(x_1-x_2)^2+(y_1-y_2)^2]^{3/2}}, \tag{2.25}$$

$$m_2\frac{d^2x_2}{dt^2} = -\frac{Gm_2m_1(x_2-x_1)}{[(x_2-x_1)^2+(y_2-y_1)^2]^{3/2}}, \quad m_2\frac{d^2y_2}{dt^2} = -\frac{Gm_2m_1(y_2-y_1)}{[(x_2-x_1)^2+(y_2-y_1)^2]^{3/2}}. \tag{2.26}$$

Choosing the center of mass of the system as the origin of the coordinates, and putting the two bodies on x axis at the initial time $t = 0$, we obtain, from Eq. (2.3),

$$m_1x_1 + m_2x_2 = 0. \tag{2.27}$$

At the initial time, we put the two bodies at the pericenter, then we have

$$\begin{cases} t = 0, \\ x_1 = a_{e1}(1-e) \equiv r_{p1}, \\ y_1 = 0, \\ x_2 = -\dfrac{m_1}{m_2}x_1 = -\dfrac{m_1}{m_2}r_{p1}, \\ y_2 = 0, \end{cases} \tag{2.28}$$

where a_{e1} denotes the semi-major axis of the first body. At a half period $t = T_\mathrm{K}/2$, where T_K is the orbital period, since the bodies are at the apocenter, the coordinate values of the two bodies are

$$\begin{cases} t' = \dfrac{T_\mathrm{K}}{2}, \\ x_1' = -\dfrac{(1+e)}{(1-e)}r_{p1}, \\ y_1' = 0, \\ x_2' = -\dfrac{m_1}{m_2}x_1' = \dfrac{m_1}{m_2}\dfrac{(1+e)}{(1-e)}r_{p1}, \\ y_2' = 0. \end{cases} \tag{2.29}$$

Since the velocity of each body is perpendicular to x axis at the pericenter and the apocenter, denoting the components of the velocity as v_{xI} and v_{yI}, respectively, the x components v_{xI} are zero. Thus, from the law of conservation of the energy, we obtain

$$\frac{1}{2}m_1 v_{y1}^2 + \frac{1}{2}m_2 v_{y2}^2 - G\frac{m_1 m_2}{r_{p1}+\frac{m_1}{m_2}r_{p1}} = \frac{1}{2}m_1 v_{y1}'^2 + \frac{1}{2}m_2 v_{y2}'^2 - G\frac{m_1 m_2}{\frac{(1+e)}{(1-e)}r_{p1}+\frac{m_1}{m_2}\frac{(1+e)}{(1-e)}r_{p1}}. \tag{2.30}$$

From the law of conservation of angular momentum, we obtain

$$v_{y1}\cdot r_{p1} = v_{y1}'\cdot\left[-\frac{(1+e)}{(1-e)}r_{p1}\right],\quad v_{y2}\cdot\left[-\frac{m_1}{m_2}r_{p1}\right] = v_{y2}'\cdot\left[\frac{m_1}{m_2}\frac{(1+e)}{(1-e)}r_{p1}\right]. \tag{2.31}$$

Differentiating Eq. (2.3) with respect to time and substituting $t=0$, we obtain $m_1 v_{y1}+m_2 v_{y2}=0$. Therefore, we have

$$v_{y1}' = -\frac{(1-e)}{(1+e)}v_{y1},\quad v_{y2} = -\frac{m_1}{m_2}v_{y1},\quad v_{y2}' = \frac{m_1}{m_2}\frac{(1-e)}{(1+e)}v_{y1}. \tag{2.32}$$

Substituting Eq. (2.32) into Eq. (2.30) and choosing the positive sign, v_{y1} can be expressed as

$$v_{y1} = \frac{m_2}{m_1+m_2}\sqrt{\frac{Gm_2(1+e)}{r_{p1}}}. \tag{2.33}$$

Figure 2.2 shows the elliptic orbits when the mass ratio is $m_1 : m_2 = 2:3$ and the eccentricity is $e=0.3$. The red and the green disks denote m_1 and m_2, respectively.

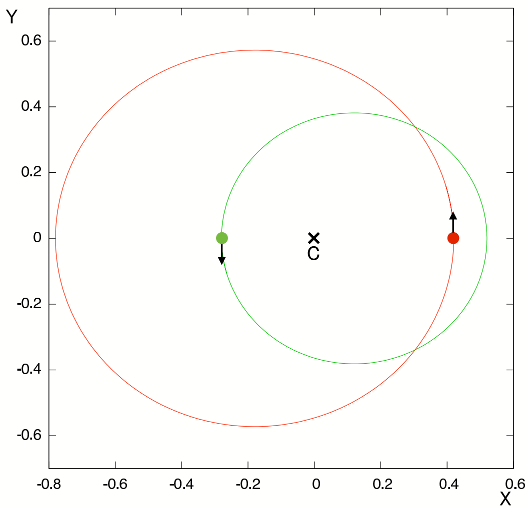

Figure 2.2: Elliptic two-body orbits.

2.2 Equations of motion for N-body systems

Since the Newton's gravitational theory is a linear theory of gravity, the principle of superposition holds. Therefore, the Lagrangian of a N-body system is

$$\mathscr{L} = \sum_I \frac{m_I \dot{\boldsymbol{r}}_I^2}{2} + \sum_I \sum_J{}' \frac{G m_I m_J}{2 r_{IJ}}, \tag{2.34}$$

where $r_{IJ} \equiv |\boldsymbol{r}_I - \boldsymbol{r}_J|$ and the prime on the summation sign means that we should omit the term with $J = I$.

The equations of motion of the Kth body is obtained from the Euler-Lagrange equation:

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{\boldsymbol{r}}_K} = \frac{\partial \mathscr{L}}{\partial \boldsymbol{r}_K}. \tag{2.35}$$

As a result, we obtain

$$m_K \frac{d^2 \boldsymbol{r}_K}{dt^2} = -\sum_{A \neq K} \frac{G m_K m_A}{r_{KA}^3} \boldsymbol{r}_{KA}, \tag{2.36}$$

where $\boldsymbol{r}_{KA} \equiv \boldsymbol{r}_K - \boldsymbol{r}_A$. For the three-body system, this becomes

$$m_K \frac{d^2 \boldsymbol{r}_K}{dt^2} = -\frac{G m_K m_I}{r_{KI}^3} \boldsymbol{r}_{KI} - \frac{G m_K m_J}{r_{KJ}^3} \boldsymbol{r}_{KJ}. \tag{2.37}$$

2.3 Equilibrium solutions to the three-body problem

In general, the three-body problem cannot be solved even in Newtonian gravity, while particular solutions are obtained [36]. As analytic solutions, Euler's collinear and Lagrange's equilateral triangular solutions are known and they are the *equilibrium solutions*, where the centrifugal force balances with the gravitational one for each body. For the restricted three-body problem, these solutions are known as Lagrangian points (see Fig. 2.3). In this section, we briefly summarize the derivations of these two solutions in circular motion.

2.3.1 Euler's collinear solution

First, we consider a collinear configuration in which the bodies always line up and move around the center of mass with the same constant angular velocity ω_{N}.

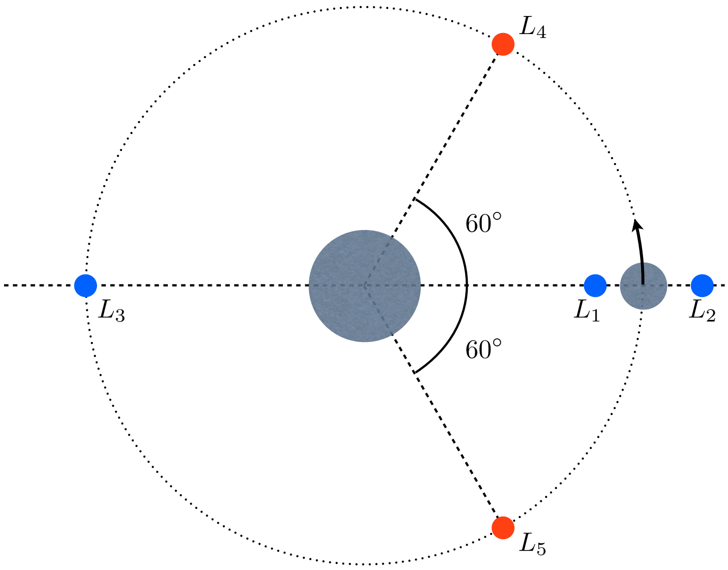

Figure 2.3: Lagrangian points around massive two bodies. The two gray disks are massive two bodies. L_1, L_2, and L_3 (blue disks) correspond to Euler's collinear solution and L_4 and L_5 (red disks) are Lagrange's equilateral triangular solution.

It is convenient to use the corotating frame with the same angular velocity. We choose the (x, y) plane as the orbital plane normal to the total angular momentum of the system in such a corotating frame, where the three bodies are located along the x coordinate. Then, the location of each body is written as $\boldsymbol{r}_I = (x_I, 0)$.

We treat the three bodies equivalently, then we can put $x_3 < x_2 < x_1$ without loss of generality. Choosing the origin of the coordinates as the center of mass, we have $x_1 > 0$ and $x_3 < 0$. The equations of motion for the three bodies in Newtonian gravity are

$$r_1\omega^2 = \frac{Gm_2}{r_{12}^2} + \frac{Gm_3}{r_{13}^2}, \tag{2.38}$$

$$r_2\omega^2 = -\frac{Gm_1}{r_{12}^2} + \frac{Gm_3}{r_{23}^2}, \tag{2.39}$$

$$r_3\omega^2 = -\frac{Gm_1}{r_{13}^2} - \frac{Gm_2}{r_{23}^2}, \tag{2.40}$$

where $r_I \equiv |\boldsymbol{r}_I|$. Figure 2.4 shows a collinear configuration in the corotating frame. The filled disks denote each body.

We define a distance ratio as $z \equiv r_{23}/r_{12}$, which is an important variable in the following

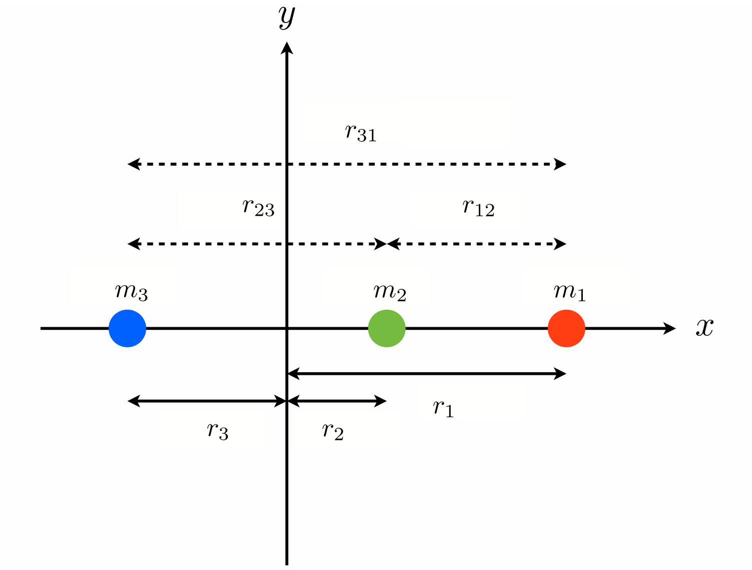

Figure 2.4: A collinear configuration.

formulation, and then, we have $r_{13} = (1+z)r_{12}$.

First, we subtract Eq. (2.39) from Eq. (2.38) and Eq. (2.40) from Eq. (2.39). Such a subtraction procedure will be useful also at the PN order because we can avoid using the expression of the PN center of mass [84, 78, 124]. Next, we compute a ratio between them to delete ω_N^2. As a result, we obtain a quintic equation

$$(m_1+m_2)z^5+(3m_1+2m_2)z^4+(3m_1+m_2)z^3-(m_2+3m_3)z^2-(2m_2+3m_3)z-(m_2+m_3)=0 \quad (2.41)$$

for the distance ratio $z > 0$. Such a quintic equation cannot be solved in algebraic manners as shown by Galois (e.g. [122]).

However, we can know the number of positive roots of Eq. (2.41). Descartes' rule of signs (e.g. [122]) states that the number of positive roots of a polynomial equation either equals that of sign changes in coefficients of the polynomial or is less than it by a multiple of two. According to this rule, the quintic equation (2.41) has the only one positive root for given masses. Substituting the positive root z into a difference between any two equations of motion, for instance between Eqs. (2.38) and (2.40), we obtain the corresponding angular velocity ω_N. It is known that Euler's solution is unstable (e.g. [36, 121]).

2.3.2 Lagrange's equilateral triangular solution

Next, Let us summarize Lagrange's solution in Newtonian gravity. We choose the origin of the coordinates such that

$$m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 + m_3\boldsymbol{r}_3 = \boldsymbol{0}. \tag{2.42}$$

We consider an equilateral triangular configuration. Then, we put

$$r_{12} = r_{23} = r_{31} = \ell. \tag{2.43}$$

The Newtonian equations of motion for each body are

$$\ddot{\boldsymbol{r}}_I = -\frac{GM}{\ell^3}\boldsymbol{r}_I, \tag{2.44}$$

where the dot means differentiation with respect to time t and $M = \sum_I m_I$ is the total mass of the system. Hence, each body can move around the center of mass with the same angular velocity $\omega_{\rm N}$ given by

$$\omega_{\rm N}^2 = \frac{GM}{\ell^3}. \tag{2.45}$$

The velocity of each body $\boldsymbol{v}_I (\equiv \dot{\boldsymbol{r}}_I)$ is perpendicular to the position vector $\boldsymbol{r}_I$ and its magnitude is

$$v_I \equiv |\boldsymbol{v}_I| = \omega_{\rm N} r_I. \tag{2.46}$$

From Eq. (2.42), the position vector of each body can be rewritten by the relative vector as

$$\boldsymbol{r}_I = \sum_J \nu_J \boldsymbol{r}_{IJ}, \tag{2.47}$$

where $\boldsymbol{r}_{IJ} \equiv \boldsymbol{r}_I - \boldsymbol{r}_J$ and we denote the mass ratio as $\nu_I = m_I/M$. Thus, the orbital radius of each body is

$$r_I = \ell\sqrt{\nu_J^2 + \nu_J\nu_K + \nu_K^2}, \tag{2.48}$$

where we have used the relation as $\boldsymbol{r}_{IJ}\cdot\boldsymbol{r}_{KI} = -\ell^2/2$ for $I \neq J \neq K$.

Figure 2.5 shows an equilateral triangular configuration. Let ϑ_{IJ} denote the angle from $\boldsymbol{r}_I$ to

$\boldsymbol{r}_J$. Using the cosine formula, it is shown that

$$\cos\vartheta_{12} = \frac{r_1^2 + r_2^2 - \ell^2}{2r_1r_2}, \ \sin\vartheta_{12} = \frac{\sqrt{3}v_3\ell^2}{2r_1r_2}, \tag{2.49}$$

$$\cos\vartheta_{23} = \frac{r_2^2 + r_3^2 - \ell^2}{2r_2r_3}, \ \sin\vartheta_{23} = \frac{\sqrt{3}v_1\ell^2}{2r_2r_3}, \tag{2.50}$$

$$\cos\vartheta_{31} = \frac{r_3^2 + r_1^2 - \ell^2}{2r_3r_1}, \ \sin\vartheta_{31} = \frac{\sqrt{3}v_2\ell^2}{2r_3r_1}. \tag{2.51}$$

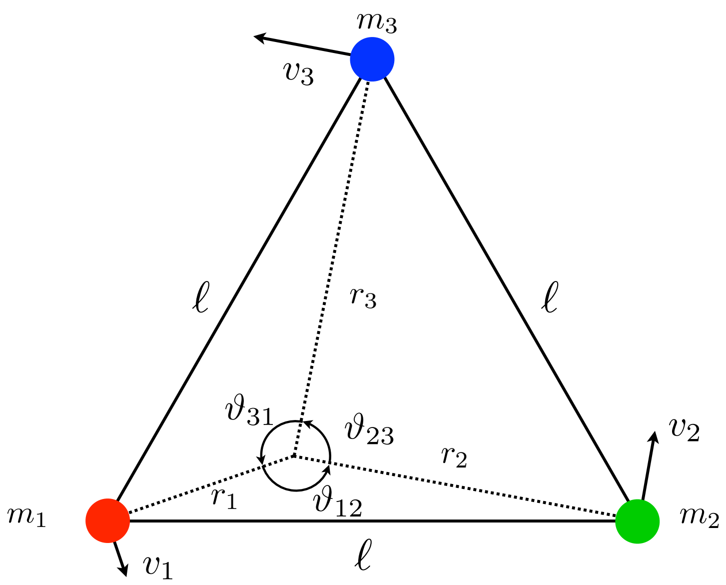

Figure 2.5: An equilateral triangular configuration.

Gascheau proved that Lagrange's equilateral triangular configuration for circular motion is stable [53], if

$$0 < \frac{m_1m_2 + m_2m_3 + m_3m_1}{(m_1 + m_2 + m_3)^2} < \frac{1}{27}. \tag{2.52}$$

Routh extended the result to a general law of gravitation $\propto 1/r^k$, and found the condition for stability as [98]

$$0 < \frac{m_1m_2 + m_2m_3 + m_3m_1}{(m_1 + m_2 + m_3)^2} < \frac{1}{3}\left(\frac{3-k}{1+k}\right)^2. \tag{2.53}$$

Chapter 3

Post-Newtonian Equations of Motion

In the framework of general relativity, the comparison of the theory with Newtonian gravity and with experiments becomes particularly simple when one takes the slow-motion, weak-field limit. Such a limit called the PN approximation is sufficiently accurate to consider the solar system. However, the PN approximation is not appropriate to discuss the system of compact objects such as binary pulsar, where the slow-motion and the weak-field assumptions are not valid.

In this chapter, we study the 1PN equations of motion for bodies that have no extent. For this, we denote the curved space-time metric by $g_{\mu\nu}$ and its determinant by g (so $g < 0$). The Christoffel symbol is

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma} \right). \tag{3.1}$$

The Riemann tensor, the Ricci tensor, and the Ricci scalar are

$$R_{\mu\nu\rho\sigma} = \frac{1}{2} \left(g_{\mu\sigma,\nu\rho} + g_{\nu\rho,\mu\sigma} - g_{\mu\rho,\nu\sigma} - g_{\nu\sigma,\mu\rho} \right) + g_{\alpha\beta} \left(\Gamma^{\alpha}_{\nu\rho} \Gamma^{\beta}_{\mu\sigma} - \Gamma^{\alpha}_{\nu\sigma} \Gamma^{\beta}_{\mu\rho} \right), \tag{3.2}$$

$$R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu} = R^{\alpha}{}_{\mu\alpha\nu} = \frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}} - \frac{\partial \Gamma^{\alpha}_{\mu\alpha}}{\partial x^{\nu}} + \Gamma^{\alpha}_{\mu\nu} \Gamma^{\beta}_{\alpha\beta} - \Gamma^{\beta}_{\mu\alpha} \Gamma^{\alpha}_{\nu\beta}, \tag{3.3}$$

$$R = g^{\mu\nu} R_{\mu\nu} = R^{\mu}_{\mu}, \tag{3.4}$$

respectively. The energy-momentum tensor $T^{\mu\nu}$ is defined from the variation of the matter action S_M under a change of the metric $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$, according to

$$\delta S_M = \frac{1}{2c} \int d^4x \, \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu}. \tag{3.5}$$

The energy-momentum tensor of a perfect fluid is expressed as

$$T^{\mu\nu} = (\mu + p)u^\mu u^\nu + pg^{\mu\nu}, \tag{3.6}$$

where μ is the energy density of matter per the 4-volume element $\sqrt{-g}d^4x$, then μ/c^2 is the mass density of matter, and p and u^μ are the pressure and the four-velocity of matter. The Einstein equations read

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{3.7}$$

where $G_{\mu\nu}$ is the Einstein tensor defined by

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R. \tag{3.8}$$

This chapter is based on Refs. [38, 39, 78, 84, 124].

3.1 The spherically symmetric gravitational field

First, we study a spherically symmetric gravitational field. Such a gravitational field can be produced by matter of any spherically symmetric distribution. For spherically symmetry, not only the matter distribution but also the matter motion must be spherically symmetric. Namely, the velocity at each point must be parallel to the radial direction. The spherically symmetric field means that the line element ds at a time must be the same for all points located at the same distance from the center. In other words, the metric tensor depends only on the time and the distance.

Using spherical coordinates (r, θ, φ), the most general expression of the spherically symmetric spacetime is

$$ds^2 = A(t,r)dt^2 + B(t,r)dtdr + C(t,r)dr^2 + D(t,r)(d\theta^2 + \sin^2\theta d\varphi^2), \tag{3.9}$$

where A, B, C, and D are functions of time t and radius r. Here, because of the arbitrariness of the choice of the reference frame in general relativity, we can perform a coordinate transformation without breaking the spherically symmetry. Thus, we can choose new coordinates t and r, for which the coefficient $B(t,r)$ vanishes and the coefficient $D(t,r)$ becomes simply r^2. The latter condition implies that the circumference of the circle of radius r is equal to $2\pi r$. Note that these conditions do not determine uniquely the choice of the time coordinate. We can perform again any transformation $t = f(\tilde{t})$, not containing r, where $\tilde{t}$ is a new time coordinate.

It is convenient to write the quantities $A(t,r)$ and $C(t,r)$ in the form of exponential functions as $-c^2e^\nu$ and e^λ, respectively, where λ and ν are functions of t and r. Thus, ds^2 can be expressed as

$$ds^2 = -e^\nu c^2 dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{3.10}$$

Denoting by x^0, x^1, x^2, and x^3 the coordinates ct, r, θ, and φ, respectively, the non-zero components of the metric tensor are

$$g_{00} = -e^\nu, \quad g_{11} = e^\lambda, \quad g_{22} = r^2, \quad g_{33} = r^2\sin^2\theta. \tag{3.11}$$

Thus,

$$g^{00} = -e^{-\nu}, \quad g^{11} = e^{-\lambda}, \quad g^{22} = r^{-2}, \quad g^{33} = r^{-2}\sin^{-2}\theta. \tag{3.12}$$

From Eq. (3.1), we obtain the expressions as

$$\begin{aligned}
&\Gamma^0_{00} = \frac{\dot{\nu}}{2}, \quad && \Gamma^0_{10} = \frac{\nu'}{2}, \quad && \Gamma^0_{11} = \frac{\dot{\lambda}}{2}e^{\lambda-\nu}, && && \\
&\Gamma^1_{00} = \frac{\nu'}{2}e^{\nu-\lambda}, \quad && \Gamma^1_{10} = \frac{\dot{\lambda}}{2}, \quad && \Gamma^1_{11} = \frac{\lambda'}{2}, \quad && \Gamma^1_{22} = -re^{-\lambda}, \quad && \Gamma^1_{33} = -r\sin^2\theta e^{-\lambda}, \\
&\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r}, \quad && \Gamma^2_{33} = -\sin\theta\cos\theta, \quad && \Gamma^3_{23} = \cot\theta, && &&
\end{aligned} \tag{3.13}$$

where the dot and the prime denote differentiation with respect to ct and r, respectively. All other components, except for those which differ from the ones we have written by transposition of the indices μ and ν, are zero.

From Eqs. (3.3) and (3.4), the non-zero components of the Ricci tensor and the Ricci scalar are

$$R_{00} = \left[\frac{1}{2}\nu'' + \frac{1}{4}\nu'^2 - \frac{1}{4}\nu'\lambda' + \frac{\nu'}{r}\right]e^{\nu-\lambda} - \frac{1}{4}\dot{\lambda}^2 + \frac{1}{4}\dot{\nu}\dot{\lambda} - \frac{1}{2}\ddot{\lambda}, \tag{3.14}$$

$$R_{01} = \frac{\dot{\lambda}}{r}, \tag{3.15}$$

$$R_{11} = \left[\frac{1}{4}\dot{\lambda}^2 - \frac{1}{4}\dot{\nu}\dot{\lambda} + \frac{1}{2}\ddot{\lambda}\right]e^{\lambda-\nu} - \frac{1}{2}\nu'' - \frac{1}{4}\nu'^2 + \frac{1}{4}\nu'\lambda' + \frac{\lambda'}{r}, \tag{3.16}$$

$$R_{22} = \left[\frac{1}{2}\lambda' r - \frac{1}{2}\nu' r - 1\right] e^{-\lambda} + 1, \tag{3.17}$$

$$R_{33} = \sin^2\theta R_{22}, \tag{3.18}$$

$$R = \left[\frac{1}{2}\dot{\nu}\dot{\lambda} - \frac{1}{2}\dot{\lambda}^2 - \ddot{\lambda}\right] e^{-\nu} + \left[\nu'' + \frac{1}{2}\nu'^2 - \frac{1}{2}\nu'\lambda' + \frac{2}{r}\nu' - \frac{2}{r}\lambda' + \frac{2}{r^2}\right] e^{-\lambda} - \frac{2}{r^2}. \tag{3.19}$$

The Einstein equations (3.7) become

$$-e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2} = \frac{8\pi G}{c^4} T^0_0, \tag{3.20}$$

$$-e^{-\lambda}\frac{\dot{\lambda}}{r} = \frac{8\pi G}{c^4} T^1_0, \tag{3.21}$$

$$-e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} = \frac{8\pi G}{c^4} T^1_1, \tag{3.22}$$

$$-\frac{1}{2}e^{-\lambda}\left(\nu'' + \frac{\nu'^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu'\lambda'}{2}\right) + \frac{1}{2}e^{-\nu}\left(\ddot{\lambda} + \frac{\dot{\lambda}^2}{2} - \frac{\dot{\lambda}\dot{\nu}}{2}\right) = \frac{8\pi G}{c^4} T^2_2 = \frac{8\pi G}{c^4} T^3_3. \tag{3.23}$$

The other components identically vanish. Using Eq. (3.6), the components of the energy-momentum tensor are expressed by the energy density μ of the matter, its pressure p, and the radial velocity $u^1 = dr/d\tau$.

Equations (3.20) - (3.23) can be integrated exactly when we consider vacuum, that is, outside of the masses producing the field. Setting the energy-momentum tensor equal to zero, we obtain the equations

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 0, \tag{3.24}$$

$$e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 0, \tag{3.25}$$

$$\dot{\lambda} = 0, \tag{3.26}$$

where the fourth equation (3.23) can be derived from the other three equations.

Equation (3.26) means that λ does not depend on time. Furthermore, adding the equations Eqs. (3.24) and (3.25), we obtain

$$\lambda + \nu = f(t), \tag{3.27}$$

where $f(t)$ is a function only of time. This claims that $\lambda + \nu$ at a certain time is the same value everywhere. However, it is still possible to transform the time coordinate. If we perform the transformation $t = F(t')$ such that $dF(t')/dt' = e^{f(t)/2}$, Eq. (3.27) becomes $\nu + \lambda = 0$ without loss of generality. From this, note that the spherically symmetric gravitational field in vacuum is static, automatically.

Equation (3.25) can be rewritten as

$$(1 - r\lambda')e^{-\lambda} = 1 \quad \Leftrightarrow \quad (re^{-\lambda})' = 1. \tag{3.28}$$

It can be integrated and we obtain

$$e^{-\lambda} = e^{\nu} = 1 + \frac{r_g}{r}, \tag{3.29}$$

where r_g is a constant of integration. Requiring the Newton's gravitational theory holds at large distances where the field is weak, this constant is determined as

$$r_g = \frac{2Gm}{c^2}, \tag{3.30}$$

where m is the total mass of the matter. This quantity r_g has the dimensions of length and is called the Schwarzschild radius. This metric at infinity becomes the Minkowski metric.

Thus, the line element is expressed as

$$ds^2 = -\left(1 - \frac{r_g}{r}\right)c^2dt^2 + \left(1 - \frac{r_g}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{3.31}$$

This solution of the Einstein equations is called the Schwarzschild solution. This expression completely determines the gravitational field in vacuum produced by any spherically symmetric distribution of matter. Note that the Schwarzschild solution is valid as long as the motion of matter has the spherical symmetry.

Finally, the approximate expression of ds^2 at large distances from the origin of the coordinate is

$$ds^2 \simeq ds_0^2 - \frac{2Gm}{c^2r}(c^2dt^2 + dr^2), \tag{3.32}$$

where

$$ds_0^2 = \eta_{\mu\nu}dx^\mu dx^\nu, \tag{3.33}$$

and the second term is a small correction.

3.2 The weak gravitational field by a single body

Let us consider the weak gravitational field produced by a spherically symmetric single body with mass m at large distances r from the body, and determine the first terms of its expansion in powers of r_g/r.

At large distances, the gravitational field is weak. Then, we can choose the reference frame at large distances where the metric is almost expressed by the Minkowski metric:

$$\eta_{00} = -1, \quad \eta_{0i} = 0, \quad \eta_{ik} = \delta_{ik}. \tag{3.34}$$

Thus, we express $g_{\mu\nu}$ as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{3.35}$$

where $h_{\mu\nu}$ is a small correction determined by the gravitational field.

Operations of raising and lowering indices of small tensor $h_{\mu\nu}$ are preformed using the Minkowski metric, e.g. $h_\mu^\nu = \eta^{\nu\rho} h_{\mu\rho}$, etc. To terms of the first order,

$$g_{\mu\rho} g^{\rho\nu} = (\eta_{\mu\rho} + h_{\mu\rho}) g^{\rho\nu} = \delta_\mu^\nu + \mathrm{O}(h^2), \tag{3.36}$$

where $h = h_\mu^\mu$. Then, we obtain the contravariant components of the metric as

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \mathrm{O}(h^2). \tag{3.37}$$

And, the determinant of the metric is expressed, to terms of the first order, as

$$g = -(1+h) + \mathrm{O}(h^2). \tag{3.38}$$

Note that the coordinates are not uniquely determined by the conditions that $h_{\mu\nu}$ are small. If these conditions are satisfied in a coordinate system, even after any gauge transformation $x'^\mu = x^\mu + \xi^\mu$ with a small quantity ξ^μ, the conditions are also satisfied. Under the transformation, the metric, according to the tensor transformation, becomes

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \simeq g_{\mu\nu}(x') + \delta g_{\mu\nu}(x'), \tag{3.39}$$

to terms of first order in ξ^μ, where

$$\delta g_{\mu\nu}(x') = -\xi_{\mu;\nu} - \xi_{\nu;\mu} \tag{3.40}$$

with $\xi_\mu = \eta_{\mu\nu}\xi^\nu$. From Eqs. (3.39) and (3.40), we obtain

$$h'_{\mu\nu} \simeq h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}. \tag{3.41}$$

Note that the covariant derivatives in Eq. (3.40) become the partial ones, since $\eta_{\mu\nu}$ are constants.

In the first approximation, to terms of order r_g/r, the small corrections to the Minkowski metric are given by the corresponding terms in the expansion of the spherically symmetric Schwarzschild metric. Expressing the Schwarzschild metric by the form of Eq. (3.31), the first terms are given by Eq. (3.32). Using the Minkowski coordinates (ct, x, y, z), we obtain, to terms of the first order,

$$h_{00} = \frac{r_g}{r} + \mathrm{O}\left(\frac{r_g^2}{r^2}\right), \quad h_{0i} = \mathrm{O}\left(\frac{r_g^2}{r^2}\right), \quad h_{ik} = \frac{r_g}{r}\delta_{ik} + \mathrm{O}\left(\frac{r_g^2}{r^2}\right). \tag{3.42}$$

3.3 The Einstein-Infeld-Hoffmann equations of motion

The loss of energy of a system in the form of radiation of GWs appears in the fifth order of approximation in v/c, where v is a typical internal velocity, for the first time [17]. [1] In this chapter, we write formally $\epsilon \equiv v/c$, henceforth. Namely, the energy of the system is constant in the fourth order in the absence of an electromagnetic field. [2] In this section, let us derive the Lagrangian of a system of gravitating bodies to terms of second order in ϵ, namely, the 1PN order. And thus, we will find the equations of motion of the bodies in the next order of the Newtonian one, Eq. (2.36). Here, we neglect the extents and internal structure of the bodies, regarding them as point-like particle. [3]

[1] See Appendix D.

[2] Since the loss of energy by dipole radiation appears in the third order in ϵ, a system is conservative in general to terms of the second order in existence of an electromagnetic field.

[3] See Ref. [61] for extended bodies, for instance.

We start with the expressions $h_{\mu\nu}$ of the form Eq. (3.42) as the weak field at large distances:

$$h_{00} = \frac{r_g}{r} + \mathrm{O}(\epsilon^4), \quad h_{0i} = \mathrm{O}(\epsilon^3), \tag{3.43}$$

$$h_{ij} = \frac{r_g}{r}\delta_{ij} + \mathrm{O}(\epsilon^4). \tag{3.44}$$

In these expressions, we assumed that the field is produced by a single body. Consider the linear order in h, then the field at large distances from the system of the bodies is given by a summation of each field:

$$h_{00} = -\frac{2}{c^2}\phi + \mathrm{O}(\epsilon^4), \quad h_{0i} = \mathrm{O}(\epsilon^3), \tag{3.45}$$

$$h_{ij} = -\frac{2}{c^2}\phi\delta_i^j + \mathrm{O}(\epsilon^4), \tag{3.46}$$

where

$$\phi(\boldsymbol{r}) \equiv -\sum_I \frac{Gm_I}{|\boldsymbol{r} - \boldsymbol{r}_I|} \tag{3.47}$$

is the Newtonian gravitational potential, $\boldsymbol{r}_I$ is the position vector to the Ith body. The line element by the metric tensor of Eqs. (3.45) and (3.46) is expressed as

$$ds^2 = -\left(1 + \frac{2}{c^2}\phi\right)c^2dt^2 + \left(1 - \frac{2}{c^2}\phi\right)\left(dx^2 + dy^2 + dz^2\right) + \mathrm{O}(\epsilon^4). \tag{3.48}$$

Note that the first order terms of ϕ appear not only in g_{00} but also g_{ij}. In the equations of motion of time-like particles, the contributions from g_{ij} are higher order than those from g_{00}.

In order to obtain the required equations of motion, it is sufficient to know the spatial components of the metric to terms of the second order in ϵ as Eq. (3.46). On the other hand, the mixed and the time components are needed to terms of the third order and the fourth order, respectively.

For the energy-momentum tensor of the bodies, we put, in Eq. (3.6), $p = 0$ and

$$\mu = \sum_I \frac{m_I c^2}{\sqrt{-g(x_I)}} \frac{d\tau_I}{dt} \delta(\boldsymbol{r} - \boldsymbol{r}_I), \tag{3.49}$$

where $d\tau_I = \sqrt{-g_{\mu\nu}(x_I)dx_I^\mu dx_I^\nu/c^2}$, $\delta(\boldsymbol{r})$ is the delta function, and the summation extends over all the bodies of the system. Then, we obtain

$$T^{\mu\nu} = \sum_I \frac{m_I}{\sqrt{-g(x_I)}} \frac{dx_I^\mu}{d\tau_I} \frac{dx_I^\nu}{dt} \delta(\boldsymbol{r} - \boldsymbol{r}_I). \tag{3.50}$$

The covariant components become

$$T_{\mu\nu} = \sum_I \frac{m_I}{\sqrt{-g(x_I)}} g_{\mu\rho}(x_I) g_{\nu\sigma}(x_I) \frac{dx_I^\rho}{d\tau_I} \frac{dx_I^\sigma}{dt} \delta(\boldsymbol{r} - \boldsymbol{r}_I). \tag{3.51}$$

Thus,

$$\begin{aligned} T_{00} &= \sum_I \frac{m_I c^2}{\sqrt{-g(x_I)}} (g_{00}(x_I))^2 \frac{dt}{d\tau_I} \delta(\boldsymbol{r} - \boldsymbol{r}_I) \\ &= \sum_I m_I c^2 \left(1 + \frac{5}{c^2}\phi_I + \frac{v_I^2}{2c^2} + \mathrm{O}(\epsilon^4)\right) \delta(\boldsymbol{r} - \boldsymbol{r}_I), \end{aligned} \tag{3.52}$$

where $v_I^i = dx_I^i/dt$ is the ordinary three-dimensional velocity and ϕ_I is the potential of the field at the point $\boldsymbol{r}_I$ of the Ith body. Note that for the last equality sign in Eq. (3.52) we have used the fact obtained from Eq. (3.48)

$$\frac{dt}{d\tau_I} = 1 - \frac{1}{c^2}\phi + \frac{1}{2c^2} v_I^2 + \mathrm{O}(\epsilon^4). \tag{3.53}$$

For the components T_{0i} and T_{ij}, it is sufficient to know the leading order terms, and they are expressed as

$$T_{0i} = \sum_I m_I c v_{Ii} \delta(\boldsymbol{r} - \boldsymbol{r}_I), \tag{3.54}$$

$$T_{ij} = \sum_I m_I v_{Ii} v_{Ij} \delta(\boldsymbol{r} - \boldsymbol{r}_I). \tag{3.55}$$

And also, we can obtain

$$T = g^{\mu\nu} T_{\mu\nu} = -\sum_I m_I c^2 \left(1 + \frac{3}{c^2}\phi_I - \frac{v_I^2}{2c^2} + \mathrm{O}(\epsilon^4)\right) \delta(\boldsymbol{r} - \boldsymbol{r}_I) \tag{3.56}$$

to terms of the second order in ϵ.

Next, let us calculate the Ricci tensor $R_{\mu\nu}$ by using Eqs. (3.2) and (3.3). R_{00} is needed to compute to terms of order ϵ^4, and it is

$$R_{00} = \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\partial h^i_0}{\partial x^i} - \frac{1}{2c}\frac{\partial h^i_i}{\partial t}\right) - \frac{1}{2}\Delta h_{00} + \frac{1}{2}h^{ij}\frac{\partial^2 h_{00}}{\partial x^i \partial x^j} - \frac{1}{4}\left(\frac{\partial h_{00}}{\partial x^i}\right)^2 + \frac{1}{4}\frac{\partial h_{00}}{\partial x^j}\left(2\frac{\partial h^i_j}{\partial x^i} - \frac{\partial h^i_i}{\partial x^j}\right) + \mathrm{O}(\epsilon^5), \tag{3.57}$$

where Δ is the Laplace operator in the flat spacetime. Note that the leading terms of h_{00} and h_{ij} are of the order of ϵ^2 and $h_{0i} = \mathrm{O}(\epsilon^3)$.

In this calculation, we have not fixed the gauge. In other words, we can impose a gauge condition

$$\frac{\partial h^i_0}{\partial x^i} - \frac{1}{2c}\frac{\partial h^i_i}{\partial t} = 0, \tag{3.58}$$

which fix one of the degrees of freedom of gauge. As a result, the components h_{0i} drop out in R_{00}. Substituting

$$h^j_i = \frac{2}{c^2}\phi\delta^j_i + \mathrm{O}(\epsilon^4), \quad h_{00} = -\frac{2}{c^2}\phi + \mathrm{O}(\epsilon^4) \tag{3.59}$$

into the remaining terms, we obtain

$$R_{00} = -\frac{1}{2}\Delta h_{00} + \frac{2}{c^4}\Delta\phi - \frac{2}{c^4}(\nabla\phi)^2 + \mathrm{O}(\epsilon^5), \tag{3.60}$$

where $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$ is the three-dimensional gradient.

For the mixed components R_{0i}, it is sufficient to know the terms of the leading order, that is, ϵ^3. As a result, we obtain

$$R_{0i} = \frac{1}{2c}\frac{\partial^2 h^j_i}{\partial t \partial x^j} + \frac{1}{2}\frac{\partial^2 h^j_0}{\partial x^i \partial x^j} - \frac{1}{2c}\frac{\partial^2 h^j_j}{\partial t \partial x^i} - \frac{1}{2}\Delta h_{0i} + \mathrm{O}(\epsilon^4). \tag{3.61}$$

Imposing the condition Eq. (3.58) and the relation Eq. (3.59), these become

$$R_{0i} = -\frac{1}{2}\Delta h_{0i} + \frac{1}{2c^3}\frac{\partial^2 \phi}{\partial t \partial x^i} + \mathrm{O}(\epsilon^4). \tag{3.62}$$

Using the expressions of Eqs. (3.52) - (3.62), we can write the Einstein equations

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right). \tag{3.63}$$

The time component of Eq. (3.63) becomes

$$-\Delta h_{00} + \frac{4}{c^4}\phi\Delta\phi - \frac{4}{c^4}(\nabla\phi)^2 = \frac{8\pi G}{c^4}\sum_I m_I c^2\left(1 + \frac{5\phi_I}{c^2} + \frac{3v_I^2}{2c^2}\right)\delta(\boldsymbol{r} - \boldsymbol{r}_I) + \mathrm{O}(\epsilon^5). \tag{3.64}$$

Using the identity

$$4(\nabla\phi)^2 = 2\Delta(\phi^2) - 4\phi\Delta\phi \tag{3.65}$$

and the Poisson's equation for the Newtonian potential

$$\Delta\phi = 4\pi G\sum_I m_I\delta(\boldsymbol{r} - \boldsymbol{r}_I), \tag{3.66}$$

this equation can be rewritten as

$$-\Delta\left(h_{00} + \frac{2}{c^4}\phi^2\right) = \frac{8\pi G}{c^2}\sum_I m_I\left(1 + \frac{\phi_I'}{c^2} + \frac{3v_I^2}{2c^2}\right)\delta(\boldsymbol{r} - \boldsymbol{r}_I) + \mathrm{O}(\epsilon^5), \tag{3.67}$$

where we replace ϕ_I on the right-hand-side of Eq. (3.67) by

$$\phi_I' = -G{\sum_J}'\frac{m_J}{|\boldsymbol{r}_I - \boldsymbol{r}_J|}, \tag{3.68}$$

i.e. by the potential at the point $\boldsymbol{r}_I$ produced by all the bodies except the Ith.

Taking into account the relation

$$\Delta\left(\frac{1}{r}\right) = -4\pi\delta(\boldsymbol{r}), \tag{3.69}$$

the solution of Eq. (3.67) can be obtain as

$$h_{00} = -\frac{2\phi}{c^2} - \frac{2\phi^2}{c^4} + \frac{2G}{c^4}\sum_I\frac{m_I\phi_I'}{|\boldsymbol{r} - \boldsymbol{r}_I|} + \frac{3G}{c^4}\sum_I\frac{m_I v_I^2}{|\boldsymbol{r} - \boldsymbol{r}_I|} + \mathrm{O}(\epsilon^5). \tag{3.70}$$

The mixed components of Eq. (3.63) become

$$\Delta h_{0i} = \frac{16\pi G}{c^3}\sum_I m_I v_{Ii}\delta(\boldsymbol{r}-\boldsymbol{r}_I) + \frac{1}{c^3}\frac{\partial^2 \phi}{\partial t \partial x^i} + \mathrm{O}(\epsilon^4). \tag{3.71}$$

The solution of this linear equation is

$$h_{0i} = -\frac{4G}{c^3}\sum_I \frac{m_I v_{Ii}}{|\boldsymbol{r}-\boldsymbol{r}_I|} + \frac{1}{c^3}\frac{\partial^2 f}{\partial t \partial x^i} + \mathrm{O}(\epsilon^4), \tag{3.72}$$

where f is the solution of the auxiliary equation

$$\Delta f = \phi = -\sum_I \frac{G m_I}{|\boldsymbol{r}-\boldsymbol{r}_I|}. \tag{3.73}$$

Using the relation $\Delta r = 2/r$, we obtain

$$f = -\frac{G}{2}\sum_I m_I |\boldsymbol{r}-\boldsymbol{r}_I|. \tag{3.74}$$

As a result, the mixed components are obtained as

$$h_{0i} = -\frac{G}{2c^3}\sum_I \frac{m_I}{|\boldsymbol{r}-\boldsymbol{r}_I|}[7v_{Ii} + (\boldsymbol{v}_I\boldsymbol{n}_I)n_{Ii}] + \mathrm{O}(\epsilon^4), \tag{3.75}$$

where $\boldsymbol{n}_I = (\boldsymbol{r}-\boldsymbol{r}_I)/|\boldsymbol{r}-\boldsymbol{r}_I|$.

In order to calculate the required Lagrangian at the 1PN order, it is sufficient to use Eqs. (3.46), (3.70), and (3.75). The Lagrangian for the Ith body in a gravitational field produced by the other bodies is

$$\mathscr{L}_I = -m_I c\frac{d\tau}{dt} = -m_I c^2\left(1 - h_{00} - 2h_{0i}\frac{v_I^i}{c} - \frac{v_I^2}{c^2} - h_{ij}\frac{v_I^i v_I^j}{c^2}\right)^{1/2} + \mathrm{O}(\epsilon^4). \tag{3.76}$$

Expanding the square root and dropping the irrelevant constant $-m_I c^2$, we obtain, to terms of the second order,

$$\mathscr{L}_I = \frac{m_I v_I^2}{2} + \frac{m_I v_I^4}{8c^2} + m_I c^2\left(\frac{h_{00}}{2} + h_{0i}\frac{v_I^i}{c} + \frac{1}{2c^2}h_{ij}v_I^i v_I^j + \frac{h_{00}^2}{8} + \frac{h_{00}}{4c^2}v_I^2\right) + \mathrm{O}(\epsilon^4). \tag{3.77}$$

Note that all the $h_{\mu\nu}$ are evaluated at the point $\boldsymbol{r}_I$.

The total Lagrangian $\mathscr{L}$ of the system is constructed so that it leads to the correct values of the forces $\boldsymbol{f}_I$ acting on each body for a given motion of the others. Therefore, the force $\boldsymbol{f}_I$ given by differentiating $\mathscr{L}_I$ is also obtained by taking the partial derivatives $\partial \mathscr{L}/\partial \boldsymbol{r}_I$:

$$\boldsymbol{f}_I = \frac{\partial \mathscr{L}_I}{\partial \boldsymbol{r}_I} = \frac{\partial \mathscr{L}}{\partial \boldsymbol{r}_I}. \tag{3.78}$$

As a result, we obtain the total Lagrangian of the system as

$$\begin{aligned} \mathscr{L} = &\sum_I \sum_J{}' \frac{G m_I m_J}{2 r_{IJ}} + \sum_I \frac{m_I v_I^2}{2} + \sum_I \sum_J{}' \frac{3 G m_I m_J v_I^2}{2c^2 r_{IJ}} + \sum_I \frac{m_I v_I^4}{8c^2} \\ &- \sum_I \sum_J{}' \frac{G m_I m_J}{4c^2 r_{IJ}} [7(\boldsymbol{v_I} \cdot \boldsymbol{v_J}) + (\boldsymbol{v}_I \cdot \boldsymbol{n}_{IJ})(\boldsymbol{v}_J \cdot \boldsymbol{n}_{IJ})] - \sum_I \sum_J{}' \sum_K{}' \frac{G^2 m_I m_J m_K}{2c^2 r_{IJ} r_{IK}}, \end{aligned} \tag{3.79}$$

where $r_{IJ} = |\boldsymbol{r}_I - \boldsymbol{r}_J|$, $\boldsymbol{n}_{IJ} = (\boldsymbol{r}_I - \boldsymbol{r}_J)/r_{IJ}$, the dot denotes the inner product of the vectors in the flat spacetime, and the prime on the summation sign means that we should omit the term with $J = I$ or $K = I$. The first and second terms in Eq. (3.79) agree with the Lagrangian in Newtonian gravity.

Finally, we calculate the equations of motion for the Kth body by substituting the total Lagrangian Eq. (3.79) into the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \boldsymbol{v}_K}\right) = \left(\frac{\partial \mathscr{L}}{\partial \boldsymbol{r}_K}\right). \tag{3.80}$$

These become

$$\begin{aligned} m_K \frac{d\boldsymbol{v}_K}{dt} = &\sum_{A \neq K} \boldsymbol{r}_{AK} \frac{G m_K m_A}{r_{AK}^3} \left[1 - 4 \sum_{B \neq K} \frac{G m_B}{c^2 r_{BK}} - \sum_{C \neq A} \frac{G m_C}{c^2 r_{CA}} \left(1 - \frac{\boldsymbol{r}_{AK} \cdot \boldsymbol{r}_{CA}}{2 r_{CA}^2} \right) \right. \\ &\left. + \left(\frac{v_K}{c}\right)^2 + 2\left(\frac{v_A}{c}\right)^2 - 4\left(\frac{\boldsymbol{v}_A}{c} \cdot \frac{\boldsymbol{v}_K}{c}\right) - \frac{3}{2}\left(\frac{\boldsymbol{v}_A}{c} \cdot \boldsymbol{n}_{AK}\right)^2 \right] \\ &- \sum_{A \neq K} (\boldsymbol{v}_A - \boldsymbol{v}_K) \frac{G m_K m_A}{c^2 r_{AK}^2} \boldsymbol{n}_{AK} \cdot (3\boldsymbol{v}_A - 4\boldsymbol{v}_K) + \frac{7}{2} \sum_{A \neq K} \sum_{C \neq A} \boldsymbol{r}_{CA} \frac{G^2 m_K m_A m_C}{c^2 r_{AK} r_{CA}^3}. \end{aligned} \tag{3.81}$$

Equation (3.81) is known as the Einstein-Infeld-Hoffmann (EIH) equation of motion [84, 124] for N-body systems at the 1PN order. In the Newtonian limit, Eq. (3.81) reduces to Eq. (2.36). For the two-body problem, the dynamics of each body was obtained by Damour and Deruelle [33] in the 1PN approximation.

For the relative position $\boldsymbol{r}_{JK} = \boldsymbol{r}_J - \boldsymbol{r}_K$, we obtain the equations of motion

$$
\begin{aligned}
\frac{d^2\boldsymbol{r}_{JK}}{dt^2} = \sum_{A\neq K} \boldsymbol{r}_{AK}\frac{Gm_A}{r_{AK}^3}&\left[1 - 4\sum_{B\neq K}\frac{Gm_B}{c^2 r_{BK}} - \sum_{C\neq A}\frac{Gm_C}{c^2 r_{CA}}\left(1 - \frac{\boldsymbol{r}_{AK}\cdot\boldsymbol{r}_{CA}}{2r_{CA}^2}\right)\right. \\
&\left. + \left(\frac{v_K}{c}\right)^2 + 2\left(\frac{v_A}{c}\right)^2 - 4\left(\frac{\boldsymbol{v}_A}{c}\cdot\frac{\boldsymbol{v}_K}{c}\right) - \frac{3}{2}\left(\frac{\boldsymbol{v}_A}{c}\cdot\boldsymbol{n}_{AK}\right)^2\right] \\
&- \sum_{A\neq J}\boldsymbol{r}_{AJ}\frac{Gm_A}{r_{AJ}^3}\left[1 - 4\sum_{B\neq J}\frac{Gm_B}{c^2 r_{BJ}} - \sum_{C\neq A}\frac{Gm_C}{c^2 r_{CA}}\left(1 - \frac{\boldsymbol{r}_{AJ}\cdot\boldsymbol{r}_{CA}}{2r_{CA}^2}\right)\right. \\
&\left. + \left(\frac{v_J}{c}\right)^2 + 2\left(\frac{v_A}{c}\right)^2 - 4\left(\frac{\boldsymbol{v}_A}{c}\cdot\frac{\boldsymbol{v}_J}{c}\right) - \frac{3}{2}\left(\frac{\boldsymbol{v}_A}{c}\cdot\boldsymbol{n}_{AJ}\right)^2\right] \\
&- \sum_{A\neq K}(\boldsymbol{v}_A - \boldsymbol{v}_K)\frac{Gm_A}{c^2 r_{AK}^2}\boldsymbol{n}_{AK}\cdot(3\boldsymbol{v}_A - 4\boldsymbol{v}_K) + \sum_{A\neq J}(\boldsymbol{v}_A - \boldsymbol{v}_J)\frac{Gm_A}{c^2 r_{AJ}^2}\boldsymbol{n}_{AJ}\cdot(3\boldsymbol{v}_A - 4\boldsymbol{v}_J) \\
&+ \frac{7}{2}\sum_{A\neq K}\sum_{C\neq A}\boldsymbol{r}_{CA}\frac{G^2 m_A m_C}{c^2 r_{AK} r_{CA}^3} - \frac{7}{2}\sum_{A\neq J}\sum_{C\neq A}\boldsymbol{r}_{CA}\frac{G^2 m_A m_C}{c^2 r_{AJ} r_{CA}^3}.
\end{aligned}
\tag{3.82}
$$

Chapter 4

A Post-Newtonian Collinear Solution

In this chapter, we discuss a PN collinear solution to the three-body problem in circular motion by employing the EIH equations of motion.

As we saw in Chapter 2, the Newtonian collinear configuration is determined through a master equation of fifth order in the distance ratio of the bodies. The quintic equation has only one physical root corresponding to the equilibrium configuration for given system parameters (the masses and the end-to-end length) [36, 54, 121].

In the framework of general relativity, this solution has been reexamined [129, 130]. In this case, the master equation determining the PN collinear configuration becomes seventh order. In the same way as the Newtonian case, the septic equation has always one physically acceptable root and only one. We shall call this the *uniqueness of the collinear solution*. Especially, for the restricted three-body problem, the uniqueness means that three equilibrium points of a test particle exist even at the 1PN order. They are relativistic counterparts of the Lagrangian points L_1, L_2, and L_3.

The purposes of this chapter are to derive the PN collinear solution in the circular motion and to prove the uniqueness of the configuration for given system parameters. This chapter is based on [129, 130].

4.1 The septic equation

In this section, we derive the septic equation that gives the collinear configuration. In order to take into account the dominant part of the general relativistic effects, we employ the EIH equations of motion, namely, Eq. (3.81). In the case of a collinear configuration and the circular

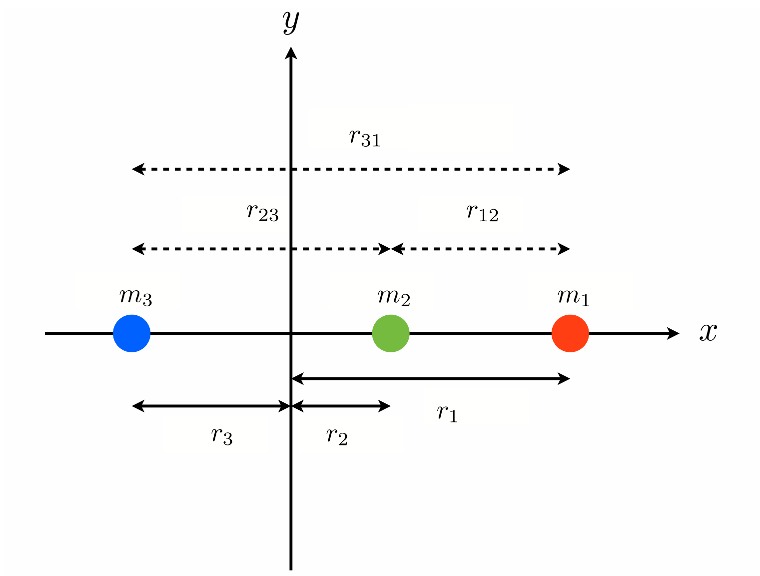

Figure 4.1: A collinear configuration.

motion, we should note that all the locations $\boldsymbol{r}_I$ of the bodies are parallel or antiparallel to each other and all the velocities $\boldsymbol{v}_I$ of the bodies are always perpendicular to $\boldsymbol{r}_I$. Hence, we obtain

$$v_I = r_I \omega, \tag{4.1}$$

where $v_I = |\boldsymbol{v}_I|$, $r_I = |\boldsymbol{r}_I|$, and ω is the common angular velocity. Figure 4.1 shows a collinear configuration in the corotating frame. The filled disks denote each body.

We define $\boldsymbol{r}_{IJ} \equiv \boldsymbol{r}_I - \boldsymbol{r}_J$ and $r_{IJ} \equiv |\boldsymbol{r}_{IJ}|$. Subtracting the EIH equation of motion for m_3 from that for m_1, for instance, and defining a distance ratio as $z \equiv r_{23}/r_{12}$, i.e. $r_{13} = (1+z)r_{12}$, we obtain

$$r_{13}\omega^2 = F_{\mathrm{N}} + \varepsilon F_M + \varepsilon F_V \omega^2, \tag{4.2}$$

where we define a non-dimensional small parameter as

$$\varepsilon \equiv \frac{GM}{c^2 \ell} \tag{4.3}$$

with the total mass $M = \sum_I m_I$ and the characteristic length of the system $\ell \equiv r_{13}$, The Newtonian term F_{N} and the PN corrections F_M (dependent on the masses only) and F_V (velocity-dependent part divided by ω^2) are defined by

$$F_{\rm N} = \frac{GM}{\ell^2 z^2}\left[(\nu_1+\nu_3)z^2+\nu_2(1+z^2)(1+z)^2\right], \tag{4.4}$$

$$\begin{aligned}F_M = &-\frac{GM}{\ell^2 z^3}[\nu_2(4-4\nu_1+\nu_3)+\nu_2(12-7\nu_1+3\nu_3)z\\ &+\nu_2(12-\nu_1+\nu_3)z^2+(8-7\nu_1-7\nu_3+8\nu_1\nu_3+3\nu_1^2+3\nu_3^2)z^3\\ &+\nu_2(12+\nu_1-\nu_3)z^4+\nu_2(12+3\nu_1-7\nu_3)z^5+\nu_2(4+\nu_1-4\nu_3)z^6\big],\end{aligned} \tag{4.5}$$

$$\begin{aligned}F_V = &\frac{\ell}{(1+z)^2 z^2}\left[-\nu_1^2\nu_2-2\nu_1\nu_2(1+\nu_1-\nu_3)z\right.\\ &+(2-2\nu_1+\nu_3+6\nu_1\nu_3-3\nu_3^2+\nu_1^3-3\nu_1^2\nu_3-3\nu_1\nu_3^2+\nu_3^3)z^2\\ &+2(1+\nu_2)(1+\nu_1+\nu_3-\nu_1^2+\nu_1\nu_3-\nu_3^2)z^3\\ &+(2+\nu_1-2\nu_3-3\nu_1^2+6\nu_1\nu_3+\nu_1^3-3\nu_1^2\nu_3-3\nu_1\nu_3^2+\nu_3^3)z^4\\ &\left.-2\nu_2\nu_3(1-\nu_1+\nu_3)z^5-\nu_2\nu_3^2z^6\right],\end{aligned} \tag{4.6}$$

respectively, where $\nu_I \equiv m_I/M$ is the mass ratio. Note that we ignore the second (and higher) order of PN contributions in this truncated calculation.

After straightforward but lengthy calculations, which are similar to the Newtonian case, we obtain a septic equation

$$F(z) \equiv \sum_{k=0}^{7} A_k z^k = 0, \tag{4.7}$$

where each coefficient A_k is defined by

$$A_7 = \varepsilon\left[-4-2(\nu_1-4\nu_3)+2(\nu_1^2+2\nu_1\nu_3-2\nu_3^2)-2\nu_1\nu_3(\nu_1+\nu_3)\right], \tag{4.8}$$

$$A_6 = 1-\nu_3+\varepsilon\left[-13-(10\nu_1-17\nu_3)+2(2\nu_1^2+8\nu_1\nu_3-\nu_3^2)+2(\nu_1^3-2\nu_1^2\nu_3-3\nu_1\nu_3^2-\nu_3^3)\right], \tag{4.9}$$

$$A_5 = 2+\nu_1-2\nu_3+\varepsilon\left[-15-(18\nu_1-5\nu_3)+4(5\nu_1\nu_3+4\nu_3^2)+6(\nu_1^3-\nu_1\nu_3^2-\nu_3^3)\right], \tag{4.10}$$

$$A_4 = 1+2\nu_1-\nu_3+\varepsilon\left[-6-2(5\nu_1+2\nu_3)-4(2\nu_1^2-\nu_1\nu_3-4\nu_3^2)+2(3\nu_1^3+\nu_1^2\nu_3-2\nu_1\nu_3^2-3\nu_3^3)\right], \tag{4.11}$$

$$A_3 = -(1-\nu_1+2\nu_3)+\varepsilon\left[6+2(2\nu_1+5\nu_3)-4(4\nu_1^2+\nu_1\nu_3-2\nu_3^2)+2(3\nu_1^3+2\nu_1^2\nu_3-\nu_1\nu_3^2-3\nu_3^3)\right], \tag{4.12}$$

$$A_2 = -(2-2\nu_1+\nu_3)+\varepsilon\left[15-(5\nu_1-18\nu_3)-4(4\nu_1^2+5\nu_1\nu_3)+6(\nu_1^3+\nu_1^2\nu_3-\nu_3^3)\right], \tag{4.13}$$

$$A_1 = -(1-\nu_1)+\varepsilon\left[13-(17\nu_1-10\nu_3)+2(\nu_1^2-8\nu_1\nu_3-2\nu_3^2)+2(\nu_1^3+3\nu_1^2\nu_3+2\nu_1\nu_3^2-\nu_3^3)\right], \tag{4.14}$$

$$A_0 = \varepsilon\left[4-2(4\nu_1-\nu_3)+2(2\nu_1^2-2\nu_1\nu_3-\nu_3^2)+2\nu_1\nu_3(\nu_1+\nu_3)\right], \tag{4.15}$$

respectively.

Equation (4.7) has an antisymmetry for exchanging indices between ν_1 and ν_3, so that, we obtain the equivalent equation by making a change $z \to 1/z$. Taking into account the antisymmetry of the locations of the bodies for exchanges between ν_1 and ν_3, the antisymmetry of Eq. (4.7) may validate the complicated form of each coefficient.

Since $\varepsilon \ll 1$, the PN corrections in each coefficient must be much smaller than the Newtonian terms. Then, they cannot change the sign of each coefficient in Eqs. (4.9) - (4.14). Therefore, in the same way as the Newtonian case, the signs of Eqs. (4.9) - (4.11) are positive ($A_6 > 0$, $A_5 > 0$, and $A_4 > 0$) and those of Eqs. (4.12) - (4.14) are negative ($A_3 < 0$, $A_2 < 0$, and $A_1 < 0$).

Using a relation $\nu_1 = 1 - \nu_2 - \nu_3$, Eq. (4.15) can be rewritten as

$$A_0 = 2\varepsilon(\nu_2 + \nu_3)(2\nu_2 + 2\nu_3 + \nu_2\nu_3), \tag{4.16}$$

which immediately leads to $A_0 > 0$. In a similar manner, we have $A_7 < 0$. [1]

Thus, the number of sign changes of the coefficients in Eq. (4.7) is necessarily three. Descartes' rule of signs indicates that the septic equation (4.7) has either one or three roots. In the case of three roots, we can easily understand that one of them corresponds to Euler's collinear solution. What are the other two roots? We shall investigate them in next section.

4.2 The uniqueness of the post-Newtonian collinear solution

Figure 4.2 shows that Eq. (4.7) has three positive roots, where we put $\nu_1 = 1/7$, $\nu_2 = 5/7$, $\nu_3 = 1/7$, and $\varepsilon = 10^{-4}$. We denote the smallest, the moderate, and the largest roots as z_S, z_R, z_L, respectively.

Table 4.1 shows the numerical values of z and $\ell\omega/c$ corresponding to Figure 4.2. The velocities corresponding to z_S and z_L are $v = O(\ell\omega) = O(c)$, which do not satisfy a slow-motion condition for the PN approximation. This can be understood by the following arguments.

Because of the antisymmetry of the septic equation (4.7) for transformation as $z \leftrightarrow 1/z$ associated with exchanges between ν_1 and ν_3, the two roots z_S and z_L must be a pair.

[1] An alternative but powerful way to see this is using the anti-symmetry of the septic equation (4.7) for exchanges between indices 1 and 3 as $\nu_1 \leftrightarrow \nu_3$ with $z \leftrightarrow 1/z$. This exchange makes a change as $A_0 \to -A_7$. Using $A_0 > 0$, we have always $A_7 < 0$.

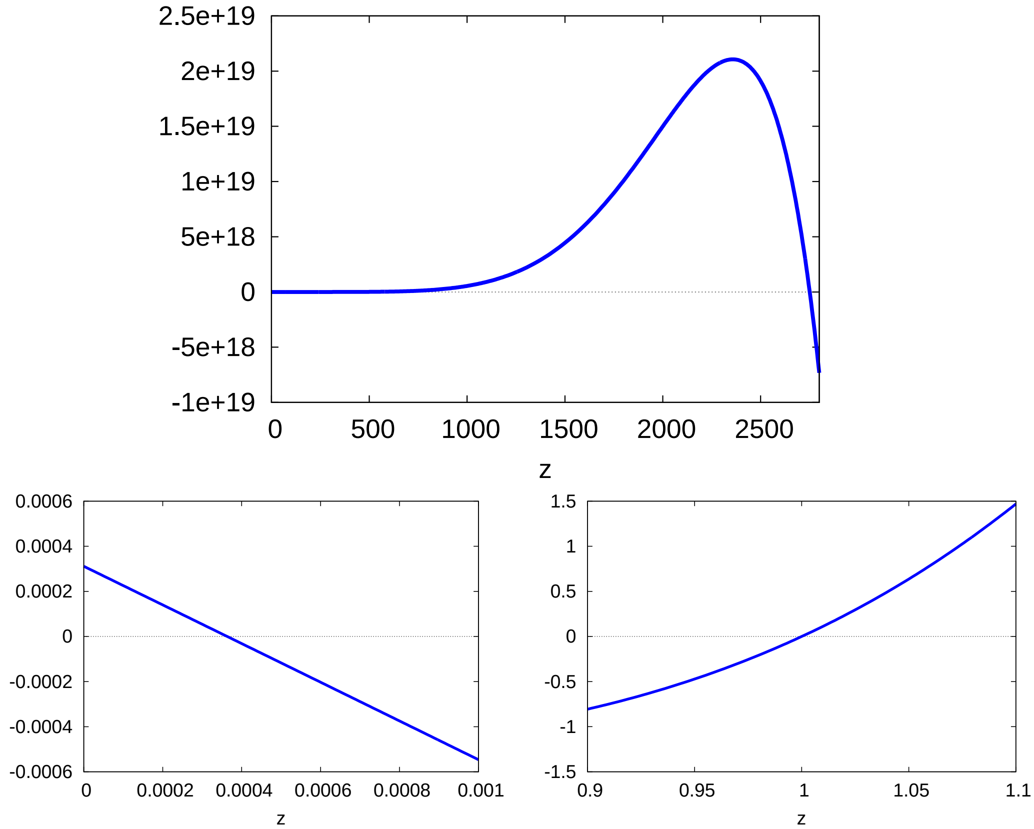

Figure 4.2: The septic polynomial. Top: the septic polynomial in Eq. (4.7). Bottom left: The polynomial around the smallest positive root. Bottom right: The polynomial around the moderate positive root

First, we consider a region where $z \ll 1$. In this region, Eq. (4.7) approximately becomes

$$A_1 z + A_0 \simeq 0, \tag{4.17}$$

where A_0 contains only the 1PN correction terms and A_1 has both the Newtonian terms $A_{\mathrm{N}1}$ and the 1PN correction terms $A_{\mathrm{PN}1}$. If the signs of A_0 and $A_{\mathrm{N}1}$ are different, then we have an approximate form of a positive root in this region as

$$z_S \simeq -\frac{A_0}{A_{\mathrm{N}1}} = \mathrm{O}(\varepsilon). \tag{4.18}$$

Table 4.1: Values of z and $\ell\omega/c$ for Figure 4.2. The velocities corresponding to z_S and z_L are $v = \mathrm{O}(\ell\omega) = \mathrm{O}(c)$, which do not satisfy a slow-motion condition for the PN approximation. This can be understood by the following arguments.

	z_S	z_R	z_L
z	3.635×10^{-4}	1.000	2751
$\ell\omega/c$	0.8723	0.02449	0.8723

Since $\varepsilon \to 0$ in the Newtonian limit, the smallest positive root z_S can appear only in the PN approximation.

Let us make an order-of-magnitude estimation using Eq. (4.2) for a velocity $\ell\omega_S$, where ω_S is the angular velocity corresponding to z_S. From the expressions of Eqs. (4.4), (4.5), and (4.6), we obtain

$$F_{\mathrm{N}} = \mathrm{O}\left(\frac{GM}{\ell^2 z_S^2}\right) = \mathrm{O}\left(\frac{GM}{\ell^2 \varepsilon^2}\right), \tag{4.19}$$

$$F_M = \mathrm{O}\left(\frac{GM}{\ell^2 z_S^3}\right) = \mathrm{O}\left(\frac{GM}{\ell^2 \varepsilon^3}\right), \tag{4.20}$$

$$F_V = \mathrm{O}\left(\frac{\ell}{z_S^2}\right) = \mathrm{O}\left(\frac{\ell}{\varepsilon^2}\right). \tag{4.21}$$

Since $\varepsilon \ll 1$, we have $r_{13} = \ell \ll \varepsilon F_V$, so that $F_{\mathrm{N}} \sim F_M \sim F_V \omega_S^2 \gg r_{13}\omega_S^2$ in Eq. (4.2). Nevertheless the Newtonian angular velocity ω_{N} is $O(\sqrt{GM/\ell^3})$, all the three bodies have an angular velocity $\omega_S = \mathrm{O}(c/\ell)$, that is,

$$\ell\omega_S = \mathrm{O}(c). \tag{4.22}$$

This means an extremely fast motion comparable to the speed of light. Such a fast motion is unacceptable in the PN approximation. Therefore, the smallest root z_S is unphysical and discarded.

Next, we consider a region where $z \gg 1$. In this region, Eq. (4.7) approximately becomes

$$A_7 z + A_6 \simeq 0, \tag{4.23}$$

where A_7 contains only the 1PN correction terms and A_6 has both the Newtonian terms $A_{\mathrm{N}6}$ and the 1PN correction terms $A_{\mathrm{PN}6}$. If the signs of A_7 and $A_{\mathrm{N}6}$ are different, then we have an approximate form of a positive root in this region as

$$z_L \simeq -\frac{A_7}{A_{\mathrm{N}6}} = \mathrm{O}\left(\frac{1}{\varepsilon}\right). \tag{4.24}$$

In the Newtonian limit as $\varepsilon \to 0$, this root disappears.

Let us make an order-of-magnitude estimation for a velocity $\ell\omega_L$, where ω_L is the angular velocity corresponding to z_L. From Eqs. (4.4), (4.5), and (4.6), we obtain

$$F_{\mathrm{N}} = \mathrm{O}\left(\frac{GMz_L^2}{\ell^2}\right) = \mathrm{O}\left(\frac{GM}{\ell^2\varepsilon^2}\right), \tag{4.25}$$

$$F_M = \mathrm{O}\left(\frac{GMz_L^3}{\ell^2}\varepsilon\right) = \mathrm{O}\left(\frac{GM}{\ell^2\varepsilon^3}\right), \tag{4.26}$$

$$F_V = \mathrm{O}\left(\ell z_L^2 \varepsilon\right) = \mathrm{O}\left(\frac{\ell}{\varepsilon^2}\right). \tag{4.27}$$

Then, $r_{13} = \ell \ll \varepsilon F_V$, so that $F_{\mathrm{N}} \sim F_M \sim F_V \omega_L^2 \gg r_{13}\omega_L^2$ in Eq. (4.2). This leads to $\omega_L = \mathrm{O}(c/\ell)$ and

$$\ell\omega_L = \mathrm{O}(c). \tag{4.28}$$

This implies that the PN approximation breaks down by such an extremely fast motion comparable to the speed of light. Therefore, the largest root z_L is unphysical and discarded.

We should remember the antisymmetry under the transformation as $z \leftrightarrow 1/z$, hence, z_S and z_L correspond to each other as $z_S \leftrightarrow 1/z_L$. In this sense, it seems natural that the above argument for discarding z_L is similar to that of z_S.

As a result, two of the three positive roots are discarded being considered as unphysical ones. Hence, we complete the proof of the uniqueness.

Substituting the physical root of Eq. (4.7) into Eq. (4.2), we obtain the angular velocity

$$\omega = \omega_{\mathrm{N}}(1 + \tilde{\omega}_{\mathrm{PN}}), \tag{4.29}$$

where $\omega_{\mathrm{N}} \equiv (F_{\mathrm{N}}/r_{13})^{1/2}$ is the Newtonian angular velocity and we denote the PN correction as

$$\tilde{\omega}_{\mathrm{PN}} = \frac{\ell F_M + F_{\mathrm{N}} F_V}{2\ell F_{\mathrm{N}}}\varepsilon \tag{4.30}$$

For the fixed masses m_I and the end-to-end length ℓ, we have always an inequality

$$\omega < \omega_{\rm N}, \tag{4.31}$$

which means that the PN orbital period measured in the coordinate time is longer than the Newtonian one. [2] If the masses and angular rate are fixed, the end-to-end length ℓ is shorter than in the Newtonian case.

Figure 4.3 shows a numerical example, where we put $m_1 : m_2 : m_3 = 1 : 2 : 3$, $r_{12} = 1$ and $\varepsilon = 0.01$. [3] In this figure, we do not use the corotating frame (x, y) but an inertial one $(\bar{x}, \bar{y})$. The disks denote the three bodies at $t = 0$, when m_1, m_2, and m_3 are located from right to left on the horizontal axis. The triangles and circles denote the locations of each body at $t = T_{\rm N}/2$ and $t = T/2$, respectively, where T and $T_{\rm N}$ are the orbital period and the Newtonian one. These are computed by using our formulations, e.g. Eq. (4.29). On the other hand, for the trajectories of the bodies, we directly solved numerically the EIH equations of motion until $t = T_{\rm N}/2$. Both methods provide the same plot. This agreement may also validate our formulation. We assume $x_3 < x_2 < x_1$ throughout this chapter. This figure suggests that as an alternative initial condition, we can assume $x_1 < x_2 < x_3$, which is realized at $t = T/2$ in this figure. This is a natural consequence of the parity symmetry in our formulation. Numerical calculations for this figure show that the PN corrections to the angular velocity in Eq. (4.29) is negative, that is $\omega < \omega_{\rm N}$ as we already mentioned above. It should be also noted that the location of each mass at $t = T/2$ is advanced compared with that at $t = T_{\rm N}/2$.

Finally, we focus on the restricted three-body problem. The physical root z_R can be expressed as $z_R = z_{\rm N}(1+\chi)$, where $z_{\rm N}$ is the corresponding Newtonian root and χ is the small correction of the order of ε. Substituting it into Eq. (4.7), we obtain the PN correction

$$\chi \simeq -\frac{\sum_k A_{{\rm PN}k} z_{\rm N}^k}{\sum_k k A_{{\rm N}k} z_{\rm N}^k}, \tag{4.32}$$

where $A_{{\rm N}k}$ and $A_{{\rm PN}k}$ are the Newtonian and 1PN terms in A_k, respectively.

For the Sun-Jupiter system, the 1PN corrections to L_1, L_2, and L_3 become $+30$, -38, $+1$ [m], respectively, where the positive sign is chosen along the direction from the Sun to Jupiter. Such corrections suggest a potential role of the general relativistic three- (or more) body dynamics in high-precision astrometry in our solar system and perhaps also in GW astronomy. They are small

[2] See Appendix B.

[3] The order of magnitude of the 1PN effects is 0.01.

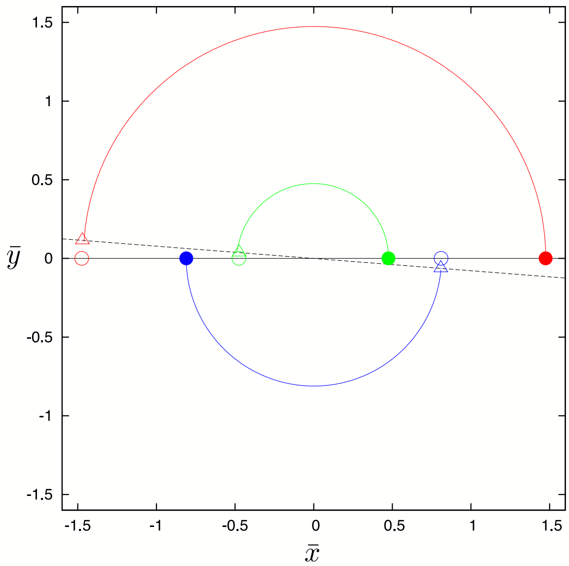

Figure 4.3: The three-body orbits for the PN collinear solution. We put $m_1 : m_2 : m_3 = 1 : 2 : 3$, $r_{12} = 1$ and $\varepsilon = 0.01$. We do not use the corotating frame (x, y) but an inertial one $(\bar{x}, \bar{y})$. The disks, triangles, and circles denote the locations of each body at $t = 0$, $t = T_N/2$, and $t = T/2$, respectively.

but may be marginal within the limits of the current technology since the lunar laser ranging experiment has successfully measured the increasing distance between the Earth and the Moon $\simeq 3.8$ cm/yr.

Chapter 5

A Post-Newtonian Triangular Solution

In this chapter, we discuss, in circular motion, a PN triangular solution to the three-body problem and its stability by employing the EIH equations of motion.

For the restricted three-body problem, a triangular equilibrium solution has been investigated by Krefetz and Maindl, independently [76, 82]. Ichita, Yamada, and Asada have studied the PN effects on Lagrange's solution for general masses [57]. Soon after, Yamada and Asada have found a PN triangular solution to the general three-body problem by adding the appropriate general relativistic corrections to the configuration of the bodies [132]. This chapter is based on [57, 132].

5.1 The post-Newtonian corrections

We take into account the dominant part of the general relativistic effect. Namely, we employ the EIH equations of motion (3.81).

First, we consider an equilateral triangular configuration. However, in this case, it is shown that a PN equilateral triangular solution does not exist except for two cases [57]: (i) three finite masses are equal and (ii) one mass is finite and the other two are zero. This means that under the assumption that the three bodies are located at the apexes of an equilateral triangle, three bodies of general masses obeying the EIH equations of motion can not move along circular orbits.

What happens by adding 1PN corrections to the equilateral triangular configuration? By *appropriate* 1PN corrections, if three bodies can circularly move without changing each relative position during motion, the general relativistic version of Lagrange's solution exists. Therefore, we seek such appropriate corrections for the equilibrium solution corresponding to the Lagrange's one at the 1PN order. For the equilateral triangular solution, since three bodies move always in a plane, the number of the degrees of freedom in the PN corrections to the configuration is two.

5.1.1 An equilateral triangular configuration

We consider three bodies in circular motion with the same angular velocity ω, so that each orbital radius r_I is a constant. Because of the circular motion, the velocity $\boldsymbol{v}_I$ of each body is perpendicular to the location $\boldsymbol{r}_I$.

In general, the expression for the location of the center of mass in the PN approximation differs from that in Newtonian gravity. However, fortunately, the expression for the center of mass remains unchanged for the equilateral triangular configuration even at the 1PN order [57]. [1] Then, the PN location $\boldsymbol{r}_I$ and orbital radius r_I of each body are unchanged from the Newtonian ones.

Let ω_I denote the angular velocity of the Ith body with PN corrections. After straightforward calculations, the EIH equations of motion for m_1 can be written as [57]

$$-\omega_1^2\boldsymbol{r}_1 = -\frac{GM}{\ell^3}\boldsymbol{r}_1 + \boldsymbol{\delta}_{\mathrm{EIH1}}\varepsilon, \tag{5.1}$$

where $M = \sum_I m_I$ is the total mass and we define the small parameter

$$\varepsilon \equiv \frac{GM}{c^2\ell}. \tag{5.2}$$

The PN term $\boldsymbol{\delta}_{\mathrm{EIH1}}$ is defined by

$$\begin{aligned}\boldsymbol{\delta}_{\mathrm{EIH1}} = \frac{1}{16}\frac{GM}{\ell^2}\frac{1}{\sqrt{\nu_2^2+\nu_2\nu_3+\nu_3^2}}\Big\{&\{16(\nu_2^2+\nu_2\nu_3+\nu_3^2)[3-(\nu_1\nu_2+\nu_2\nu_3+\nu_3\nu_1)]\\ &+9\nu_2\nu_3[2(\nu_2+\nu_3)+\nu_2^2+4\nu_2\nu_3+\nu_3^2]\}\boldsymbol{n}_1+3\sqrt{3}\nu_2\nu_3(\nu_2-\nu_3)(5-3\nu_1)\boldsymbol{n}_{\perp 1}\Big\},\end{aligned} \tag{5.3}$$

where $\boldsymbol{n}_I \equiv \boldsymbol{r}_I/r_I$ and $\boldsymbol{n}_{\perp I} \equiv \boldsymbol{v}_I/|\boldsymbol{v}_I|$. The equations of motion for m_2 and m_3 can be obtained by the cyclic manipulations of the indices as $1 \to 2 \to 3 \to 1$. From these equations of motion, we can see that an equilateral triangular solution for general masses does not exist in general, since the vector terms perpendicular to the position vector terms remain on the right-hand side of the equations of motion.

[1] See Appendix A.

The PN equilibrium configurations can be realized if and only if the following conditions (a) and (b) hold: (a) the coefficients of $\boldsymbol{n}_{\perp I}$ in the equations of motion for each body are zero and (b) the angular velocity for each body is the same in order to keep the distance between the bodies unchanged.

From Eq. (5.3) for m_1, the coefficient of $\boldsymbol{n}_{\perp 1}$ vanishes if either of ν_2 or ν_3 is zero or $\nu_2 = \nu_3$. In the same way for m_2 and m_3, the condition (a) holds in only the three cases: (i) three finite masses are equal, (ii) one mass is finite and the other two are zero, and (iii) two of the masses are finite and equal, and the third one is zero.

In the case (i), the three bodies have the same PN corrections, so that the condition (b) holds. In the case (ii), the body with finite mass rests and the other two test bodies has the equal PN corrections, thus the condition (b) holds. However, in the case (iii), for instance, putting $\nu_1 = \nu_2 = 1/2$ and $\nu_3 = 0$, we have

$$|\boldsymbol{\delta}_{\mathrm{EIH1}}| = |\boldsymbol{\delta}_{\mathrm{EIH2}}| = \frac{11}{8}\frac{GM}{\ell^2}, \tag{5.4}$$

$$|\boldsymbol{\delta}_{\mathrm{EIH3}}| = \left(\frac{63 + 11\sqrt{3}}{8}\right)\frac{GM}{\ell^2}. \tag{5.5}$$

Using the expression (2.48), the angular velocities of the bodies are

$$\omega_1 = \omega_2 = \omega_{\mathrm{N}}\left(1 - \frac{11}{32}\varepsilon\right), \tag{5.6}$$

$$\omega_3 = \omega_{\mathrm{N}}\left(1 - \frac{21\sqrt{3} + 11}{32}\varepsilon\right), \tag{5.7}$$

and thus the condition (b) does not hold.

As a result, the equilateral triangular solution realizes at the 1PN order in only two cases: (i) three finite masses are equal and (ii) one mass is finite and the other two are zero.

Note that not only an equilateral triangle but also an isosceles triangle can be acceptable in the case (ii). This is because that the three-body problem reduces to two problems of the test body around massive one, which are the PN versions of the two-body problem discussed in Chapter 2 .

5.1.2 A post-Newtonian inequilateral triangular configuration

Next, we consider a PN *inequilateral* triangular configuration with general relativistic corrections to each side length of an equilateral triangle, so that the distances between the bodies are parameterized as

$$r_{IJ} = \ell(1+\rho_{IJ}), \tag{5.8}$$

where ρ_{IJ} is dimensionless PN corrections, and $\rho_{JI} = \rho_{IJ}$. Figure 5.1 shows a PN triangular configuration.

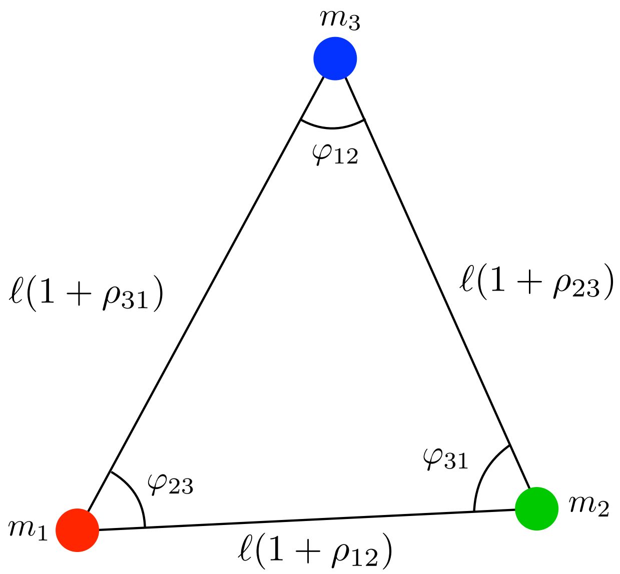

Figure 5.1: The PN corrections to the distances between the bodies.

Here, if all the three corrections are equal, that is

$$\rho_{12} = \rho_{23} = \rho_{31} = \rho, \tag{5.9}$$

then a PN configuration is still an equilateral triangle, though each side length is changed by a scale transformation as $\ell \to \ell(1+\rho)$. Namely, one of the degrees of freedom in the PN corrections corresponds to a scale transformation. However, such a scale transformation is equivalent to changing the Newtonian angular velocity as $\omega_{\mathrm{N}} \to \omega_{\mathrm{N}}(1+\rho)^{-3/2}$, so that this causes quantitative difference of the motion. [2] Therefore, this is unimportant to seek an equilibrium configuration.

[2] By the scale transformation, the 1PN terms in the equations of motion also change, such as $\boldsymbol{\delta}_{\mathrm{EIH1}}\varepsilon \to \boldsymbol{\delta}_{\mathrm{EIH1}}\varepsilon(1+\rho)^{-3}$ in Eq. (5.1). However, such changes give contributions of the second (or higher) order. Thus, it can be neglected in our consideration.

In order to eliminate this degree of freedom, we impose a constraint condition

$$\frac{r_{12}+r_{23}+r_{31}}{3}=\ell, \tag{5.10}$$

which means that the arithmetical mean of the three distances between the bodies is not changed by the PN corrections. Namely,

$$\rho_{12}+\rho_{23}+\rho_{31}=0. \tag{5.11}$$

At the 1PN order, we can employ other two conditions, which are equivalent to the above condition: [3] (i) the geometric mean of the three distances between the bodies are not changed by the PN corrections and (ii) the area of a triangle is not changed by the PN corrections. Furthermore, alternative choices of conditions are possible. For instance, (a) the angular velocity of the bodies is not changed from the Newtonian one or (b) r_{12} is not changed from the Newtonian case. In these choice, we can obtain inequilateral triangles being similar to each other.

The EIH equations of motion for m_1 are changed by the PN corrections as

$$\begin{aligned}
-\omega_1^2\boldsymbol{r}_1 &= \frac{Gm_2}{r_{21}^3}\boldsymbol{r}_{21}+\frac{Gm_3}{r_{31}^3}\boldsymbol{r}_{31}+\boldsymbol{\delta}_{\mathrm{EIH1}}\varepsilon \\
&= -\frac{GM}{\ell^3}\boldsymbol{r}_1-\frac{3}{2}\frac{GM}{\ell^3}\frac{r_1}{\nu_2^2+\nu_2\nu_3+\nu_3^2} \\
&\quad \times\{[\nu_2(\nu_1-\nu_2-1)\rho_{12}+\nu_3(\nu_1-\nu_3-1)\rho_{31}]\boldsymbol{n}_1+\sqrt{3}\nu_2\nu_3(\rho_{12}-\rho_{31})\boldsymbol{n}_{\perp 1}\} \\
&\quad +\boldsymbol{\delta}_{\mathrm{EIH1}}\varepsilon,
\end{aligned} \tag{5.12}$$

where $\boldsymbol{\delta}_{\mathrm{EIH1}}$ is expressed as Eq. (5.3). We should note that $\boldsymbol{r}_I$ may be different from the Newtonian case, because of the PN corrections. However, we can replace $\boldsymbol{r}_I$ with the Newtonian location because of the following reason. The two terms including $\boldsymbol{r}_I$ in Eq. (5.12), that is, the left hand side and the first term of the right-hand side can be expanded as

$$-\omega_1^2\boldsymbol{r}_1=-\omega_1^2\boldsymbol{r}_{\mathrm{N}1}-\omega_{\mathrm{N}}^2\boldsymbol{r}_{\mathrm{PN}1}, \tag{5.13}$$

$$-\frac{GM}{\ell^3}\boldsymbol{r}_1=-\frac{GM}{\ell^3}\boldsymbol{r}_{\mathrm{N}1}-\frac{GM}{\ell^3}\boldsymbol{r}_{\mathrm{PN}1}, \tag{5.14}$$

where $\boldsymbol{r}_{\mathrm{N}1}$ and $\boldsymbol{r}_{\mathrm{PN}1}$ are the Newtonian location and the PN correction, respectively, and $\omega_{\mathrm{N}}=$

[3] See also Appendix in Ref. [132].

$\sqrt{GM/\ell^3}$ is the Newtonian angular velocity. [4] The relation between Eqs. (5.13) and (5.14) implies that the PN corrections to $\boldsymbol{r}_I$ cancel out in Eq. (5.12).

Also, $\boldsymbol{n}_1$ and $\boldsymbol{n}_{\perp 1}$ have PN corrections. However, these corrections give contributions of the second (or higher) order in Eq. (5.12), thus they are neglected in the 1PN approximation. Also in $\boldsymbol{\delta}_{\mathrm{EIH1}}$, the PN corrections to $\boldsymbol{n}_1$ and $\boldsymbol{n}_{\perp 1}$ are the second order. We can obtain the equations of motion for m_2 and m_3 by the cyclic manipulations as $1 \to 2 \to 3 \to 1$.

The PN triangular configuration becomes an equilibrium solution in the circular motion if and only if the following conditions (a) and (b) simultaneously hold: (a) the centrifugal force balances with the gravitational force for each body and (b) the PN triangular configuration does not change with time. The condition (a) is equivalent to (a') that the coefficients of $\boldsymbol{n}_{\perp I}$ in the EIH equation of motion for each body vanish:

$$\rho_{12} - \rho_{31} - \frac{1}{8}(\nu_2 - \nu_3)(5 - 3\nu_1)\varepsilon = 0, \tag{5.15}$$

$$\rho_{23} - \rho_{12} - \frac{1}{8}(\nu_3 - \nu_1)(5 - 3\nu_2)\varepsilon = 0, \tag{5.16}$$

$$\rho_{31} - \rho_{23} - \frac{1}{8}(\nu_1 - \nu_2)(5 - 3\nu_3)\varepsilon = 0. \tag{5.17}$$

The condition (b) can be restated as: (b') the angular velocities of the three bodies are the same in order not to change the distances between the three bodies:

$$\omega_1^2 - \omega_2^2 = 0, \tag{5.18}$$

$$\omega_1^2 - \omega_3^2 = 0. \tag{5.19}$$

Equations (5.18) and (5.19) are rewritten as

$$\begin{aligned}
&\frac{3}{2}\frac{1}{\nu_2^2+\nu_2\nu_3+\nu_3^2}[\nu_2(\nu_1-\nu_2-1)\rho_{12}+\nu_3(\nu_1-\nu_3-1)\rho_{31}] \\
&-\frac{3}{2}\frac{1}{\nu_3^2+\nu_3\nu_1+\nu_1^2}[\nu_3(\nu_2-\nu_3-1)\rho_{23}+\nu_1(\nu_2-\nu_1-1)\rho_{12}] \\
&-\left\{\frac{9}{16}\frac{1}{\nu_2^2+\nu_2\nu_3+\nu_3^2}\nu_2\nu_3[2(\nu_2+\nu_3)+\nu_2^2+4\nu_2\nu_3+\nu_3^2]\right\}\varepsilon \\
&+\left\{\frac{9}{16}\frac{1}{\nu_3^2+\nu_3\nu_1+\nu_1^2}\nu_3\nu_1[2(\nu_3+\nu_1)+\nu_3^2+4\nu_3\nu_1+\nu_1^2]\right\}\varepsilon = 0,
\end{aligned} \tag{5.20}$$

[4] See Eq. (2.45)

$$
\begin{aligned}
&\frac{3}{2}\frac{1}{\nu_2^2+\nu_2\nu_3+\nu_3^2}[\nu_2(\nu_1-\nu_2-1)\rho_{12}+\nu_3(\nu_1-\nu_3-1)\rho_{31}] \\
&-\frac{3}{2}\frac{1}{\nu_1^2+\nu_1\nu_2+\nu_2^2}[\nu_1(\nu_3-\nu_1-1)\rho_{31}+\nu_2(\nu_3-\nu_2-1)\rho_{23}] \\
&-\left\{\frac{9}{16}\frac{1}{\nu_2^2+\nu_2\nu_3+\nu_3^2}\nu_2\nu_3[2(\nu_2+\nu_3)+\nu_2^2+4\nu_2\nu_3+\nu_3^2]\right\}\varepsilon \\
&+\left\{\frac{9}{16}\frac{1}{\nu_1^2+\nu_1\nu_2+\nu_2^2}\nu_1\nu_2[2(\nu_1+\nu_2)+\nu_1^2+4\nu_1\nu_2+\nu_2^2]\right\}\varepsilon=0, \qquad (5.21)
\end{aligned}
$$

respectively.

Since the number of degrees of freedom in the PN corrections (ρ_{12}, ρ_{23}, ρ_{31}) is two, it seems that the above five conditions of Eqs. (5.15) - (5.19) can not be simultaneously satisfied by any set of the PN corrections. However, the number of independent conditions turns out to be only two. This can be shown as follows.

First, we can obtain Eq. (5.17) from Eqs. (5.15) and (5.16) by eliminating ρ_{12}. Next, the left-hand sides of Eqs. (5.20) and (5.21) vanish if and only if Eqs. (5.15) and (5.16) are satisfied.

As a result, we obtain the PN corrections satisfying the above conditions as

$$\rho_{12}=\frac{1}{24}[(\nu_2-\nu_3)(5-3\nu_1)-(\nu_3-\nu_1)(5-3\nu_2)]\varepsilon, \qquad (5.22)$$

$$\rho_{23}=\frac{1}{24}[(\nu_3-\nu_1)(5-3\nu_2)-(\nu_1-\nu_2)(5-3\nu_3)]\varepsilon, \qquad (5.23)$$

$$\rho_{31}=\frac{1}{24}[(\nu_1-\nu_2)(5-3\nu_3)-(\nu_2-\nu_3)(5-3\nu_1)]\varepsilon, \qquad (5.24)$$

which give a PN triangular equilibrium solution for general masses. As $\nu_3\to 0$, these corrections reduce to the previous results for the restricted three-body problem [76, 82].

Substituting Eqs. (5.22) and (5.24) into Eq. (5.12), we obtain the common angular velocity of the three bodies

$$\omega=\omega_{\mathrm{N}}(1+\tilde{\omega}_{\mathrm{PN}}), \qquad (5.25)$$

where we denote the PN correction as

$$\tilde{\omega}_{\mathrm{PN}}=-\frac{1}{16}[29-14(\nu_1\nu_2+\nu_2\nu_3+\nu_3\nu_1)]\varepsilon. \qquad (5.26)$$

Taking into account that $\nu_1+\nu_2+\nu_3=1$, we can show $\tilde{\omega}_{\mathrm{PN}}<0$, that is, $\omega<\omega_{\mathrm{N}}$ for the fixed system parameters ℓ and ν_I. In other words, the area of the PN triangular configuration is always smaller than in the Newtonian case if the masses and angular velocity of the three bodies are fixed.

Table 5.1 shows the PN corrections to the distances between the bodies and the Lagrangian point L_4 (L_5) in the solar system. Here we choose the bodies of m_1 and m_2 as the Sun and each planet, respectively, and then $v_3 = 0$. For the solar system, it is not natural to change r_{12} but location of the test body because the distance between the Sun and each planet has been estimated. [5] Therefore, we perform a scale transformation as $\ell \to \ell(1+\rho_{12})$. By the scale transformation, the PN corrections to r_{23} and r_{31} are changed as $\rho_{23} \to \rho_{23} - \rho_{12}$ and $\rho_{31} \to \rho_{31} - \rho_{12}$, and hence it is convenient to use Eqs. (5.15) and (5.16) rather than Eqs. (5.22) - (5.24). From Eq. (5.16), we can obtain the PN correction to the distance between each planet and the Lagrangian point L_4 (L_5) as

$$\ell(\rho_{23} - \rho_{12}) = -\frac{5}{16}\frac{2GM_\odot}{c^2}\left(1 - \frac{3}{5}\frac{M_{\mathrm{P}}}{M}\right), \tag{5.27}$$

where M_{P} is the mass of planet. Equation (5.27) shows that the correction to the distance between each planet and the Lagrangian point L_4 (L_5) becomes approximately 5/16 of the Schwarzschild radius of the Sun and the planetary contribution is much smaller. Thus, we obtain almost the same values for this correction for Earth and Jupiter. The similar corrections are mentioned also in the previous paper [82].

Table 5.1: The PN corrections to the Lagrangian point L_4 (L_5) of the solar system.

Planet	Sun-L_4 (L_5) [m]	Planet-L_4 (L_5) [m]
Jupiter	-0.353	-923
Earth	-0.00111	-923

[5]The distance between the Sun and Jupiter, for instance, has been estimated from the orbital period of Jupiter. Therefore, it may be more natural to fix the scale of the system by the orbital period of Jupiter. For this, see Ref. [128].

5.2 A linear stability of the post-Newtonian triangular solution

Next, we study a stability of the PN triangular solution by taking account of linear perturbations in the orbital plane. Because of three bodies, the number of degrees of freedom in the perturbations is $2 \times 3 = 6$. However, two of the degrees of freedom correspond to perturbations in the center of mass. It is convenient to use the corotating coordinates with the center of mass as the origin even after adding perturbations. Hence, the number of the degrees of freedom in the perturbations decreases from six to four: one of them corresponds to a perturbation in the common angular velocity, and the three other perturbations denote changes in the shape and size of the PN triangle.

5.2.1 Equations of motion for perturbations

We consider four perturbations in the orbital plane. See Fig. 5.2 for a schematic figure of the four perturbations.

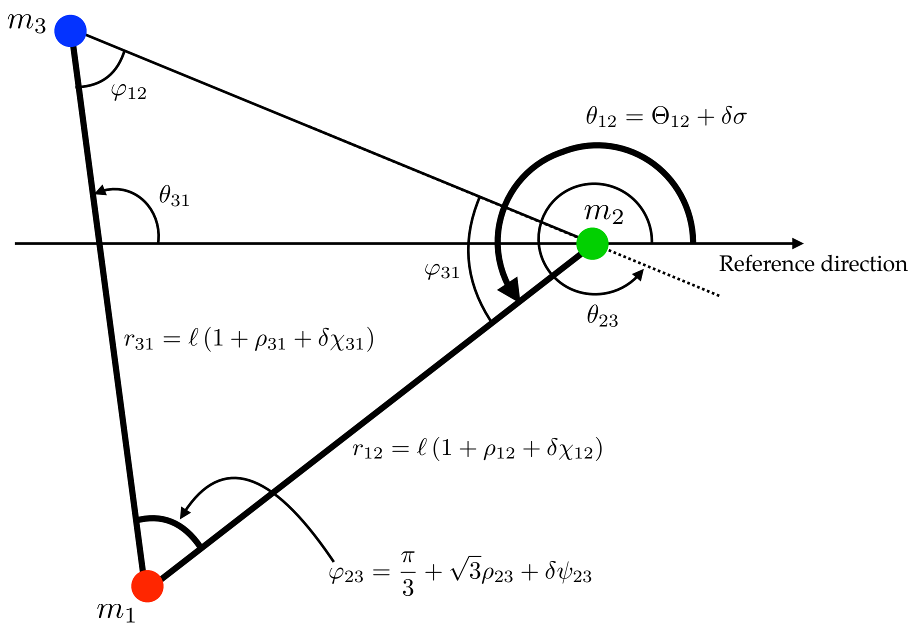

Figure 5.2: Four perturbations in the PN triangular configuration.

First, we put the distances between the bodies as

$$r_{IJ} = \ell(1 + \rho_{IJ} + \delta\chi_{IJ}), \tag{5.28}$$

where $\chi_{IJ}(= \chi_{JI})$ is a perturbation in the distance r_{IJ} and δ is a bookkeeping parameter that denotes order of smallness of the perturbations. By these perturbations, each angle φ_{IJ} between $\boldsymbol{r}_{KI}$ and $\boldsymbol{r}_{JK}$ $(I \neq J \neq K)$ of the PN triangle is changed as

$$\varphi_{IJ} \simeq \frac{\pi}{3} + \sqrt{3}\rho_{IJ} + \delta\psi_{IJ}. \tag{5.29}$$

The perturbations χ_{IJ} and ψ_{IJ} relate to each other through the cosine formula. As a result, we obtain the relation as

$$\psi_{12} = \frac{1}{\sqrt{3}}\left[2\chi_{12} - \chi_{23} - \chi_{31} + 2\rho_{12}\chi_{12} + (\rho_{23} + 4\rho_{31})\chi_{23} + (4\rho_{23} + \rho_{31})\chi_{31}\right], \tag{5.30}$$

$$\psi_{23} = \frac{1}{\sqrt{3}}\left[2\chi_{23} - \chi_{31} - \chi_{12} + 2\rho_{23}\chi_{23} + (\rho_{31} + 4\rho_{12})\chi_{31} + (4\rho_{31} + \rho_{12})\chi_{12}\right], \tag{5.31}$$

$$\psi_{31} = \frac{1}{\sqrt{3}}\left[2\chi_{31} - \chi_{12} - \chi_{23} + 2\rho_{31}\chi_{31} + (\rho_{12} + 4\rho_{23})\chi_{12} + (4\rho_{12} + \rho_{23})\chi_{23}\right]. \tag{5.32}$$

Hence, the number of independent perturbations $(\chi_{12}, \chi_{23}, \chi_{31}, \psi_{12}, \psi_{23}, \psi_{31})$ is three.

The remaining one of the degrees of freedom corresponds to a change in the common angular velocity of the bodies:

$$\theta_{IJ} = \Theta_{IJ} + \delta\sigma_{IJ}, \tag{5.33}$$

where θ_{IJ} and σ_{IJ} denote the angle from a reference direction to $\boldsymbol{r}_{IJ}$ and a perturbation in it, respectively. Θ_{IJ} is the unperturbed direction which satisfies the following equation

$$\frac{d\Theta_{IJ}}{dt} = \omega_{\mathrm{N}}(1 + \tilde{\omega}_{\mathrm{PN}}), \tag{5.34}$$

where $\tilde{\omega}_{\mathrm{PN}}$ is defined by Eq. (5.26), namely,

$$\tilde{\omega}_{\mathrm{PN}} = -\frac{1}{16}[29 - 14(\nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1)]\varepsilon. \tag{5.35}$$

Differentiating the relations

$$\theta_{23} = \theta_{12} + \pi - \varphi_{31}, \tag{5.36}$$

$$\theta_{31} = \theta_{12} - \pi + \varphi_{23}, \tag{5.37}$$

we obtain

$$D\sigma_{23} = D(\sigma - \psi_{31}), \tag{5.38}$$

$$D\sigma_{31} = D(\sigma + \psi_{23}), \tag{5.39}$$

where D denotes a differential operator with respect to normalized time $\tilde{t} \equiv \omega_{\mathrm{N}} t$ and we have denoted σ_{12} simply as σ. Equations (5.38) and (5.39) show that the perturbations σ_{12}, σ_{23}, and σ_{31} relate to each other. Thus, the number of degrees of freedom in $(\sigma_{12}, \sigma_{23}, \sigma_{31})$ is one and it corresponds to a change in the angular velocity. Note that the perturbation σ changes not only the common angular velocity but also the directions of the bodies. However, the constant part of σ can be always canceled out by the coordinates rotation. Therefore, its differential with respect to time $D\sigma$ has the physical meaning and this is the perturbation in the common angular velocity.

Note that the perturbations have not only the Newtonian terms but also the 1PN ones. For instance, the perturbation σ can be expanded as

$$\sigma = \sigma_{\mathrm{N}} + \sigma_{\mathrm{PN}}, \tag{5.40}$$

where σ_{N} and $\sigma_{\mathrm{PN}} (= \mathrm{O}(\varepsilon))$ are the Newtonian term and the 1PN one, respectively. In the following, we neglect the terms of second (and higher) order in δ. Namely, we calculate to the terms of order $\varepsilon \times \delta$ (i.e. the 1PN corrections to the linear perturbations).

Using a complex plane as the orbital one, we denote the relative position of the bodies as $\boldsymbol{r}_{IJ} \to z_{IJ} = r_{IJ} e^{i\theta_{IJ}}$. From Eq. (3.82), the EIH equation of motion for z_{12} becomes

$$\frac{d^2 z_{12}}{dt^2} = F_{12} e^{i\theta_{12}}. \tag{5.41}$$

The left-hand side of this equation is

$$\begin{aligned} \frac{d^2 z_{12}}{dt^2} = \ell \omega_{\mathrm{N}}^2 \big[&-\{1 + 2\tilde{\omega}_{\mathrm{PN}} + \rho_{12} + \delta(2D\sigma_{12} + \chi_{12} - D^2\chi_{12} + 2\tilde{\omega}_{\mathrm{PN}} D\sigma_{12} + 2\rho_{12} D\sigma_{12} \\ &+ 2\tilde{\omega}_{\mathrm{PN}}\chi_{12})\} + i\delta(2D\chi_{12} + D^2\sigma_{12} + 2\tilde{\omega}_{\mathrm{PN}} D\chi_{12} + \rho_{12} D^2\sigma_{12}) \big] e^{i\theta_{12}}. \end{aligned} \tag{5.42}$$

F_{12} in the right-hand side of the equation of motion can be expanded as

$$F_{12} = F_{\mathrm{N}\,12} + \varepsilon F_{\mathrm{PN}\,12} + \delta F_{\mathrm{Nper}\,12} + \varepsilon\delta F_{\mathrm{PNper}\,12}, \tag{5.43}$$

where $F_{\mathrm{N}\,12}$ and $F_{\mathrm{PN}\,12}$ are the unperturbed Newtonian and PN terms, respectively, and $F_{\mathrm{Nper}\,12}$ and $F_{\mathrm{PNper}\,12}$ are the perturbed Newtonian and PN terms, respectively. These are

$$\begin{aligned}
F_{\mathrm{N}\,12} &= -\frac{M}{\ell^2}, &(5.44)\\
F_{\mathrm{PN}\,12} &= \frac{1}{24}\frac{M}{\ell^2}(45\nu_2^2 + 54\nu_1\nu_2 - 60\nu_2 + 45\nu_1^2 - 60\nu_1 + 97), &(5.45)\\
F_{\mathrm{Nper}\,12} &= \frac{1}{2}\frac{M}{\ell^2}\left[3\nu_3(\chi_{23} + \chi_{31}) + 2(2 - 3\nu_3)\chi_{12}\right] + i\frac{3\sqrt{3}}{2}\frac{M}{\ell^2}\nu_3(\chi_{31} - \chi_{23}), &(5.46)\\
F_{\mathrm{PNper}\,12} &= \frac{1}{16}\frac{M}{\ell^2}\Big[-2(54\nu_2^3 + 108\nu_1\nu_2^2 - 86\nu_2^2 + 108\nu_1^2\nu_2 - 82\nu_1\nu_2 + 167\nu_2 + 54\nu_1^3\\
&\quad - 86\nu_1^2 + 167\nu_1 - 29)\chi_{12} - \nu_3(45\nu_2^2 + 108\nu_1\nu_2 + 8\nu_2 + 90\nu_1^2 - 108\nu_1 + 150)\\
&\quad \times \chi_{23} - \nu_3(90\nu_2^2 + 108\nu_1\nu_2 - 108\nu_2 + 45\nu_1^2 + 8\nu_1 + 150)\chi_{31} + 8(\nu_2^3 - \nu_1\nu_2^2\\
&\quad + 2\nu_2^2 - \nu_1^2\nu_2 - 4\nu_1\nu_2 - 7\nu_2 + \nu_1^3 + 2\nu_1^2 - 7\nu_1)D\sigma_{12} - 2\nu_3(\nu_2^2 + 22\nu_1\nu_2 + 4\nu_2\\
&\quad + 4\nu_1^2 + 2\nu_1 + 8)D\sigma_{23} - 2\nu_3(4\nu_2^2 + 22\nu_1\nu_2 + 2\nu_2 + \nu_1^2 + 4\nu_1 + 8)D\sigma_{31}\\
&\quad + 8\sqrt{3}\nu_3(\nu_1 - \nu_2)(3 - \nu_3)D\chi_{12} - 2\sqrt{3}\nu_3(9\nu_2^2 - 4\nu_1\nu_2 - 4\nu_2 + 4\nu_1^2 + 6\nu_1\\
&\quad - 16)D\chi_{23} + 2\sqrt{3}\nu_3(4\nu_2^2 - 4\nu_1\nu_2 + 6\nu_2 + 9\nu_1^2 - 4\nu_1 - 16)D\chi_{31}\Big]\\
&\quad + i\frac{1}{16}\frac{M}{\ell^2}\Big[-12\sqrt{3}\nu_3(\nu_1 - \nu_2)(3 - \nu_3)\chi_{12} + \sqrt{3}\nu_3(57\nu_2^2 + 36\nu_1\nu_2 - 24\nu_2\\
&\quad + 42\nu_1^2 - 12\nu_1 + 130)\chi_{23} - \sqrt{3}\nu_3(42\nu_2^2 + 36\nu_1\nu_2 - 12\nu_2 + 57\nu_1^2 - 24\nu_1\\
&\quad + 130)\chi_{31} - 8\sqrt{3}\nu_3(\nu_1 - \nu_2)(\nu_1 + \nu_2)D\sigma_{12} + 2\sqrt{3}\nu_3(\nu_2^2 - 12\nu_1\nu_2 + 14\nu_2\\
&\quad - 4\nu_1^2 + 10\nu_1 + 8)D\sigma_{23} + 2\sqrt{3}\nu_3(4\nu_2^2 + 12\nu_1\nu_2 - 10\nu_2 - \nu_1^2 - 14\nu_1 - 8)D\sigma_{31}\\
&\quad - 8(3\nu_2^3 + 9\nu_1\nu_2^2 - 6\nu_2^2 + 9\nu_1^2\nu_2 - 8\nu_1\nu_2 - 5\nu_2 + 3\nu_1^3 - 6\nu_1^2 - 5\nu_1)D\chi_{12}\\
&\quad - 2\nu_3(9\nu_2^2 + 30\nu_1\nu_2 + 10\nu_2 + 12\nu_1^2 - 18\nu_1 - 16)D\chi_{23} - 2\nu_3(12\nu_2^2 + 30\nu_1\nu_2\\
&\quad - 18\nu_2 + 9\nu_1^2 + 10\nu_1 - 16)D\chi_{31}\Big]. &(5.47)
\end{aligned}$$

We can obtain the EIH equations of motion for z_{23} and z_{31} by the cyclic manipulations as $1 \to 2 \to 3 \to 1$. Since the unperturbed terms in the equations give the PN triangular equilibrium solution, we focus on the perturbed terms. It is convenient to transform the variables as [98]

$$\chi_{23} = \frac{1}{2}\left[(1-3\rho_{12})\chi_{31} + (1-3\rho_{31})\chi_{12} + \sqrt{3}(1-\rho_{23})\psi_{23}\right], \tag{5.48}$$

$$X \equiv \chi_{31} - \chi_{12}. \tag{5.49}$$

As a result, we obtain the equation of motion for $\boldsymbol{r}_{12}$. Its radial part is

$$\begin{aligned}
&\left[(D^2-3)\chi_{12} - 2D\sigma - \frac{9}{4}\nu_3 X - \frac{3\sqrt{3}}{4}\nu_3\psi_{23}\right] + \varepsilon\left[-\frac{1}{32}\Big\{4\sqrt{3}(\nu_1-\nu_2)(7-9\nu_3)\nu_3 D\right.\\
&+(36\nu_2^3+234\nu_1\nu_2^2-146\nu_2^2+261\nu_1^2\nu_2-488\nu_1\nu_2+155\nu_2+63\nu_1^3-155\nu_1^2+137\nu_1\\
&-585)\Big\}\chi_{12} - \frac{1}{24}(27\nu_2^3+135\nu_1\nu_2^2-21\nu_2^2+135\nu_1^2\nu_2-210\nu_1\nu_2+24\nu_2+27\nu_1^3-21\nu_1^2\\
&+24\nu_1-155)D\sigma - \frac{1}{32}\nu_3\Big\{4\sqrt{3}(9\nu_1\nu_2+10\nu_2+9\nu_1^2-6\nu_1-4)D-(216\nu_2^2+288\nu_1\nu_2\\
&-154\nu_2+171\nu_1^2-38\nu_1+420)\Big\}X + \frac{1}{32}\nu_3\Big\{4(18\nu_2^2+27\nu_1\nu_2-2\nu_2+9\nu_1^2+14\nu_1\\
&\left.-12)D+\sqrt{3}(51\nu_2^2+114\nu_1\nu_2+2\nu_2+87\nu_1^2-120\nu_1+155)\Big\}\psi_{23}\right] = 0,
\end{aligned} \tag{5.50}$$

and the tangential part is

$$\begin{aligned}
&\left[2D\chi_{12} + D^2\sigma - \frac{3\sqrt{3}}{4}\nu_3 X + \frac{9}{4}\nu_3\psi_{23}\right] + \varepsilon\left[-\frac{1}{32}\Big\{4(9\nu_2^3+45\nu_1\nu_2^2+9\nu_2^2+45\nu_1^2\nu_2\right.\\
&-30\nu_1\nu_2-18\nu_2+9\nu_1^3+9\nu_1^2-18\nu_1+61)D+3\sqrt{3}\nu_3(12\nu_2^2-6\nu_1\nu_2+14\nu_2-15\nu_1^2\\
&+4\nu_1-5)\Big\}\chi_{12} - \frac{1}{24}\Big\{(3\nu_2^2+12\nu_1\nu_2-18\nu_2+3\nu_1^2-18\nu_1+10)D^2-3\sqrt{3}(\nu_1-\nu_2)\\
&\times\nu_3(9\nu_2+9\nu_1+4)D\Big\}\sigma + \frac{1}{32}\nu_3\Big\{4(18\nu_2^2+27\nu_1\nu_2+8\nu_2+9\nu_1^2+16\nu_1-12)D\\
&+\sqrt{3}(36\nu_2^2+72\nu_1\nu_2-54\nu_2+81\nu_1^2-90\nu_1+160)\Big\}X + \frac{1}{32}\nu_3\Big\{4\sqrt{3}(9\nu_1\nu_2+8\nu_2\\
&\left.+9\nu_1^2-4)D-9(21\nu_2^2+14\nu_1\nu_2-10\nu_2+13\nu_1^2-8\nu_1+45)\Big\}\psi_{23}\right] = 0.
\end{aligned} \tag{5.51}$$

In the same way, we obtain the equation of motion for $\boldsymbol{r}_{31}$ and its radial and tangential parts are

$$
\left[(D^2-3)\chi_{12}-2D\sigma+\left(D^2-3+\frac{9}{4}v_2\right)X-\left(2D+\frac{3\sqrt{3}}{4}v_2\right)\psi_{23}\right]+\varepsilon\left[-\frac{1}{32}\left\{4\sqrt{3}\right.\right.
$$

$$
\times(v_3-v_1)(7-9v_2)v_2D+(36v_3^3+234v_1v_3^2-146v_3^2+261v_1^2v_3-488v_1v_3+155v_3
$$

$$
\left.+63v_1^3-155v_1^2+137v_1-585)\right\}\chi_{12}-\frac{1}{24}(27v_3^3+135v_1v_3^2-21v_3^2+135v_1^2v_3-210v_1v_3
$$

$$
+24v_3+27v_1^3-21v_1^2+24v_1-155)D\sigma-\frac{1}{32}\left\{4\sqrt{3}v_2(9v_3^2+9v_1v_3+8v_3-4v_1-4)D\right.
$$

$$
-(180v_3^3+270v_1v_3^2-224v_3^2+198v_1^2v_3+8v_1v_3+419v_3+108v_1^3-54v_1^2+321v_1
$$

$$
\left.+165)\right\}X+\frac{1}{96}\left\{4(27v_3^3-39v_3^2-27v_1^2v_3+165v_1v_3-54v_3+36v_1^2-102v_1+191)D\right.
$$

$$
\left.\left.+3\sqrt{3}v_2(51v_3^2+114v_1v_3+2v_3+87v_1^2-120v_1+155)\right\}\psi_{23}\right]=0 \tag{5.52}
$$

and

$$
\left[2D\chi_{12}+D^2\sigma+\left(2D-\frac{3\sqrt{3}}{4}v_2\right)X+\left(D^2-\frac{9}{4}v_2\right)\psi_{23}\right]+\varepsilon\left[-\frac{1}{32}\left\{4(9v_3^3+45v_1v_3^2\right.\right.
$$

$$
+9v_3^2+45v_1^2v_3-30v_1v_3-18v_3+9v_1^3+9v_1^2-18v_1+61)D-3\sqrt{3}v_2(12v_3^2-6v_1v_3
$$

$$
\left.+14v_3-15v_1^2+4v_1-5)\right\}\chi_{12}-\frac{1}{24}\left\{(3v_3^2+12v_1v_3-18v_3+3v_1^2-18v_1+10)D^2\right.
$$

$$
\left.-3\sqrt{3}(v_3-v_1)(13-9v_2)v_2D\right\}\sigma+\frac{1}{32}\left\{4(9v_3^3-19v_3^2-9v_1^2v_3+27v_1v_3-2v_3-2v_1^2\right.
$$

$$
\left.-10v_1-49)D+\sqrt{3}(72v_3^2+54v_1v_3-12v_3+36v_1^2-78v_1+145)v_2\right\}X-\frac{1}{96}\left\{4(3v_3^2\right.
$$

$$
+12v_1v_3-18v_3+3v_1^2-18v_1+10)D^2-12\sqrt{3}(9v_3^2+9v_1v_3+12v_3-4v_1-4)v_2D
$$

$$
\left.\left.-27(21v_3^2+14v_1v_3-10v_3+13v_1^2-8v_1+45)v_2\right\}\psi_{23}\right]=0. \tag{5.53}
$$

5.2.2 The condition for stability in the Newtonian limit

First, we study the condition for stability in Newtonian gravity. In the Newtonian limit $\varepsilon \to 0$, by defining new variables $\dot{X} \equiv DX$, $\dot{\chi}_{12} \equiv D\chi_{12}$, $\dot{\sigma} \equiv D\sigma$, and $\dot{\psi}_{23} \equiv D\psi_{23}$, the equations of motion (5.50) - (5.53) for the perturbations are rearranged as

$$
D\boldsymbol{X} = \mathscr{M}_{\mathrm{N}}\boldsymbol{X}, \tag{5.54}
$$

where $\boldsymbol{X} \equiv (\dot{X}, \dot{\chi}_{12}, \dot{\sigma}, \dot{\psi}_{23}, X, \chi_{12}, \sigma, \psi_{23})$ and

$$\mathcal{M}_{\mathrm{N}} \equiv \begin{pmatrix} 0 & 0 & 0 & 2 & 3\left(1-\frac{3}{4}(\nu_2+\nu_3)\right) & 0 & 0 & \frac{3\sqrt{3}}{4}(\nu_2-\nu_3) \\ 0 & 0 & 2 & 0 & \frac{9}{4}\nu_3 & 3 & 0 & \frac{3\sqrt{3}}{4}\nu_3 \\ 0 & -2 & 0 & 0 & \frac{3\sqrt{3}}{4}\nu_3 & 0 & 0 & -\frac{9}{4}\nu_3 \\ -2 & 0 & 0 & 0 & \frac{3\sqrt{3}}{4}(\nu_2-\nu_3) & 0 & 0 & \frac{9}{4}(\nu_2+\nu_3) \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.55}$$

The eigenvalue equation for the coefficient matrix $\mathcal{M}_N$ is

$$\lambda^2(\lambda^2+1)\left(\lambda^2+\frac{1+\sqrt{1-27V}}{2}\right)\left(\lambda^2+\frac{1-\sqrt{1-27V}}{2}\right)=0, \tag{5.56}$$

or

$$\tau(\tau+1)\left(\tau+\frac{1+\sqrt{1-27V}}{2}\right)\left(\tau+\frac{1-\sqrt{1-27V}}{2}\right)=0, \tag{5.57}$$

where $\tau \equiv \lambda^2$, λ is the eigenvalue, and $V = \nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1$. [6] The roots of Eq. (5.57) are

$$\tau_0 = 0, \quad \tau_1 = -1, \quad \tau_2 = -\frac{1+\sqrt{1-27V}}{2}, \quad \tau_3 = -\frac{1-\sqrt{1-27V}}{2}. \tag{5.58}$$

For a positive real number a, we consider four cases:

(i) if $1/27 < V \le 1/3$, then $\tau_2 = -(1+\mathrm{i}\sqrt{a})/2$ and $\tau_3 = -(1-\mathrm{i}\sqrt{a})/2$, so that the eigenvalues are $\lambda_{0\pm} = 0$, $\lambda_{1\pm} = \pm\mathrm{i}$, $\lambda_{2\pm} = \pm(p-\mathrm{i}q)$, and $\lambda_{3\pm} = \pm(p+\mathrm{i}q)$, where $p = \sqrt{\sqrt{1+a}-1}/2$ and $q = \sqrt{\sqrt{1+a}+1}/2$.

[6] Using the relation $\nu_3 = 1-\nu_1-\nu_2$, V can be rewritten as

$$V = \frac{1}{3} - \frac{1}{4}(\nu_1-\nu_2)^2 - \frac{3}{4}\left(\nu_1+\nu_2-\frac{2}{3}\right)^2.$$

Therefore, the maximum value of V is 1/3 with $\nu_1 = \nu_2 = \nu_3 = 1/3$.

(ii) If $V = 1/27$, then $\tau_2 = \tau_3 = -1/2$, so that the eigenvalues are $\lambda_{0\pm} = 0$, $\lambda_{1\pm} = \pm\mathrm{i}$, and $\lambda_{2\pm} = \lambda_{3\pm} = \pm\mathrm{i}\sqrt{1/2}$.

(iii) If $0 < V < 1/27$, then $\tau_2 = -(1+\sqrt{a})/2 < 0$ and $\tau_3 = -(1-\sqrt{a})/2 < 0$, so that the eigenvalues are $\lambda_{0\pm} = 0$, $\lambda_{1\pm} = \pm\mathrm{i}$, $\lambda_{2\pm} = \pm\mathrm{i}\sqrt{(1+\sqrt{a})/2}$, and $\lambda_{3\pm} = \pm\mathrm{i}\sqrt{(1-\sqrt{a})/2}$.

(iv) If $V = 0$, then $\tau_2 = -1$ and $\tau_3 = 0$, so that the eigenvalues are $\lambda_{0\pm} = \lambda_{3\pm} = 0$ and $\lambda_{1\pm} = \lambda_{2\pm} = \pm\mathrm{i}$.

We consider the Jordan normal form of $\mathscr{M}_{\mathrm{N}}$ as $\boldsymbol{J} \equiv \boldsymbol{Q}^{-1}\mathscr{M}_{\mathrm{N}}\boldsymbol{Q}$, where $\boldsymbol{Q}$ is the transition matrix. Defining the new vector as $\boldsymbol{Y} \equiv \boldsymbol{Q}^{-1}\boldsymbol{X}$, Eq. (5.54) can be rewritten as

$$\boldsymbol{D}\boldsymbol{Y} = \boldsymbol{J}\boldsymbol{Y}. \tag{5.59}$$

In the case (i), we obtain

$$\boldsymbol{J} \equiv \begin{pmatrix} \mathrm{i} & 0 & & & & & & \\ & -\mathrm{i} & 0 & & & \mathbf{0} & & \\ & & 0 & 1 & & & & \\ & & & 0 & 0 & & & \\ & & & & p+q\mathrm{i} & 0 & & \\ & & & & & p-q\mathrm{i} & 0 & \\ & \mathbf{0} & & & & & -p+q\mathrm{i} & 0 \\ & & & & & & & -p-q\mathrm{i} \end{pmatrix}, \tag{5.60}$$

and

$$\boldsymbol{Q} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & * & * & * & * \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & * & * & * & * \\ \mathrm{i} & -\mathrm{i} & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ -\frac{\mathrm{i}}{2} & \frac{\mathrm{i}}{2} & 0 & -\frac{2}{3} & * & * & * & * \\ 1 & 1 & 1 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}, \tag{5.61}$$

where we denote

$$p = \frac{\sqrt{\sqrt{27V}-1}}{2}, q = \frac{\sqrt{\sqrt{27V}+1}}{2}, \tag{5.62}$$

and $*$ means a non-zero value. Thus, we have

$$\boldsymbol{Y} = (C_1 e^{\mathrm{i}\tilde{t}}, C_2 e^{-\mathrm{i}\tilde{t}}, C_3 + C_4\tilde{t}, C_4, C_5 e^{(p+q\mathrm{i})\tilde{t}}, C_6 e^{(p-q\mathrm{i})\tilde{t}}, C_7 e^{(-p+q\mathrm{i})\tilde{t}}, C_8 e^{(-p-q\mathrm{i})\tilde{t}}), \tag{5.63}$$

where C_i $(i = 1, 2, \cdots, 8)$ is a integral constant. From $\boldsymbol{X} = \boldsymbol{Q}\boldsymbol{Y}$, we obtain

$$X = C_5 Q_{55} e^{(p+q\mathrm{i})\tilde{t}} + C_6 Q_{56} e^{(p-q\mathrm{i})\tilde{t}} + C_7 Q_{57} e^{(-p+q\mathrm{i})\tilde{t}} + C_8 Q_{58} e^{(-p-q\mathrm{i})\tilde{t}}, \tag{5.64}$$

$$\begin{aligned} \chi_{12} = &-\frac{\mathrm{i}}{2} C_1 e^{\mathrm{i}\tilde{t}} + \frac{\mathrm{i}}{2} C_2 e^{\mathrm{i}\tilde{t}} - \frac{2}{3} C_4 + C_5 Q_{65} e^{(p+q\mathrm{i})\tilde{t}} + C_6 Q_{66} e^{(p-q\mathrm{i})\tilde{t}} + C_7 Q_{67} e^{(-p+q\mathrm{i})\tilde{t}} \\ &+ C_8 Q_{68} e^{(-p-q\mathrm{i})\tilde{t}}, \end{aligned} \tag{5.65}$$

$$\begin{aligned} \sigma = &C_1 e^{\mathrm{i}\tilde{t}} + C_2 e^{-\mathrm{i}\tilde{t}} + (C_3 + C_4\tilde{t}) + C_5 Q_{75} e^{(p+q\mathrm{i})\tilde{t}} + C_6 Q_{76} e^{(p-q\mathrm{i})\tilde{t}} + C_7 Q_{77} e^{(-p+q\mathrm{i})\tilde{t}} \\ &+ C_8 Q_{78} e^{(-p-q\mathrm{i})\tilde{t}}, \end{aligned} \tag{5.66}$$

$$\psi_{23} = C_5 e^{(p+q\mathrm{i})\tilde{t}} + C_6 e^{(p-q\mathrm{i})\tilde{t}} + C_7 e^{(-p+q\mathrm{i})\tilde{t}} + C_8 e^{(-p-q\mathrm{i})\tilde{t}}, \tag{5.67}$$

where Q_{ij} $(1 \le i, j \le 8)$ is the (i, j) comportment of Q. Since all the perturbations increase with time, the Lagrange's solution is unstable when $1/27 < V \le 1/3$.

In the case (ii), we obtain

$$\boldsymbol{J} \equiv \begin{pmatrix} \mathrm{i} & 0 & & & & & & \\ & -\mathrm{i} & 0 & & & & \mathbf{0} & \\ & & 0 & 1 & & & & \\ & & & 0 & 0 & & & \\ & & & & \frac{\mathrm{i}}{\sqrt{2}} & 1 & & \\ & & & & & \frac{\mathrm{i}}{\sqrt{2}} & 0 & \\ & \mathbf{0} & & & & & -\frac{\mathrm{i}}{\sqrt{2}} & 1 \\ & & & & & & & -\frac{\mathrm{i}}{\sqrt{2}} \end{pmatrix}, \tag{5.68}$$

$$
Q \equiv \begin{pmatrix}
0 & 0 & 0 & 0 & * & 0 & * & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & * & * & * & * \\
\mathrm{i} & -\mathrm{i} & 0 & 1 & * & * & * & * \\
0 & 0 & 0 & 0 & \frac{\mathrm{i}}{\sqrt{2}} & * & -\frac{\mathrm{i}}{\sqrt{2}} & * \\
0 & 0 & 0 & 0 & * & * & * & * \\
-\frac{\mathrm{i}}{2} & \frac{\mathrm{i}}{2} & 0 & -\frac{2}{3} & * & * & * & * \\
1 & 1 & 1 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1
\end{pmatrix}, \tag{5.69}
$$

where $*$ means a non-zero value. Thus, we have

$$
\boldsymbol{Y} = (C_1 e^{\mathrm{i}\tilde{t}}, C_2 e^{-\mathrm{i}\tilde{t}}, C_3 + C_4\tilde{t}, C_4, y_5, y_6, y_7, y_8 z), \tag{5.70}
$$

where

$$
y_5 \equiv (C_5 + C_6\tilde{t})e^{\mathrm{i}\tilde{t}/\sqrt{2}}, \quad y_6 \equiv C_6 e^{\mathrm{i}\tilde{t}/\sqrt{2}}, \quad y_7 \equiv (C_7 + C_8\tilde{t})e^{-\mathrm{i}\tilde{t}/\sqrt{2}}, \quad y_8 \equiv C_8 e^{-\mathrm{i}\tilde{t}/\sqrt{2}} \tag{5.71}
$$

and C_i $(i = 1, 2, \cdots, 8)$ is a integral constant. From $\boldsymbol{X} = Q\boldsymbol{Y}$, we obtain

$$
X = (C_5 + C_6\tilde{t})Q_{55}e^{\mathrm{i}\tilde{t}/\sqrt{2}} + C_6 Q_{56} e^{\mathrm{i}\tilde{t}/\sqrt{2}} + (C_7 + C_8\tilde{t})Q_{57}e^{-\mathrm{i}\tilde{t}/\sqrt{2}} + C_8 Q_{58} e^{-\mathrm{i}\tilde{t}/\sqrt{2}}, \tag{5.72}
$$

$$
\begin{aligned}
\chi_{12} = &-\frac{\mathrm{i}}{2}C_1 e^{\mathrm{i}\tilde{t}} + \frac{\mathrm{i}}{2}C_2 e^{\mathrm{i}\tilde{t}} - \frac{2}{3}C_4 + (C_5 + C_6\tilde{t})Q_{65}e^{\mathrm{i}\tilde{t}/\sqrt{2}} + C_6 Q_{66} e^{\mathrm{i}\tilde{t}/\sqrt{2}} \\
&+ (C_7 + C_8\tilde{t})Q_{67}e^{-\mathrm{i}\tilde{t}/\sqrt{2}} + C_8 Q_{68} e^{-\mathrm{i}\tilde{t}/\sqrt{2}},
\end{aligned} \tag{5.73}
$$

$$
\begin{aligned}
\sigma = &\, C_1 e^{\mathrm{i}\tilde{t}} + C_2 e^{-\mathrm{i}\tilde{t}} + (C_3 + C_4 t) + (C_5 + C_6\tilde{t})Q_{75}e^{\mathrm{i}\tilde{t}/\sqrt{2}} + C_6 Q_{76} e^{\mathrm{i}\tilde{t}/\sqrt{2}} \\
&+ (C_7 + C_8\tilde{t})Q_{77}e^{-\mathrm{i}\tilde{t}/\sqrt{2}} + C_8 Q_{78} e^{-\mathrm{i}\tilde{t}/\sqrt{2}},
\end{aligned} \tag{5.74}
$$

$$
\psi_{23} = (C_5 + C_6\tilde{t})e^{\mathrm{i}\tilde{t}/\sqrt{2}} + C_6 e^{\mathrm{i}\tilde{t}/\sqrt{2}} + (C_7 + C_8\tilde{t})e^{-\mathrm{i}\tilde{t}/\sqrt{2}} + C_8 e^{-\mathrm{i}\tilde{t}/\sqrt{2}}, \tag{5.75}
$$

where Q_{ij} $(1 \le i, j \le 8)$ is the (i, j) comportment of Q. Since all the perturbations increase with time, the Lagrange's solution is unstable if $V = 1/27$.

In the case (iii), we have

$$J \equiv \begin{pmatrix} \mathrm{i} & 0 & & & & & & \\ & -\mathrm{i} & 0 & & & \mathbf{0} & & \\ & & 0 & 1 & & & & \\ & & & 0 & 0 & & & \\ & & & & \alpha & 0 & & \\ & & & & & -\alpha & 0 & \\ & & \mathbf{0} & & & & \beta & 0 \\ & & & & & & & -\beta \end{pmatrix}, \tag{5.76}$$

and

$$Q \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & * & * & * & * \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & * & * & * & * \\ \mathrm{i} & -\mathrm{i} & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ -\frac{\mathrm{i}}{2} & \frac{\mathrm{i}}{2} & 0 & -\frac{2}{3} & * & * & * & * \\ 1 & 1 & 1 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}, \tag{5.77}$$

where we denote

$$\alpha = \mathrm{i}\sqrt{\frac{1+\sqrt{1-27V}}{2}}, \quad \beta = \mathrm{i}\sqrt{\frac{1-\sqrt{1-27V}}{2}}, \tag{5.78}$$

and $*$ means a non-zero value. Thus, we obtain

$$\boldsymbol{Y} = (C_1 e^{\mathrm{i}\tilde{t}}, C_2 e^{-\mathrm{i}\tilde{t}}, C_3 + C_4\tilde{t}, C_4, C_5 e^{\alpha\tilde{t}}, C_6 e^{-\alpha\tilde{t}}, C_7 e^{\beta\tilde{t}}, C_8 e^{-\beta\tilde{t}}), \tag{5.79}$$

where C_i $(i = 1, 2, \cdots, 8)$ is a integral constant. From $\boldsymbol{X} = Q\boldsymbol{Y}$, we have

$$X = C_5 Q_{55} e^{\alpha\tilde{t}} + C_6 Q_{56} e^{-\alpha\tilde{t}} + C_7 Q_{57} e^{\beta\tilde{t}} + C_8 Q_{58} e^{-\beta\tilde{t}}, \tag{5.80}$$

$$\chi_{12} = -\frac{\mathrm{i}}{2} C_1 e^{\mathrm{i}\tilde{t}} + \frac{\mathrm{i}}{2} C_2 e^{\mathrm{i}\tilde{t}} - \frac{2}{3} C_4 + C_5 Q_{65} e^{\alpha\tilde{t}} + C_6 Q_{66} e^{-\alpha\tilde{t}} + C_7 Q_{67} e^{\beta\tilde{t}} + C_8 Q_{68} e^{-\beta\tilde{t}}, \tag{5.81}$$

$$\sigma = C_1 e^{\mathrm{i}\tilde{t}} + C_2 e^{-\mathrm{i}\tilde{t}} + C_3 + C_4\tilde{t} + C_5 Q_{75} e^{\alpha\tilde{t}} + C_6 Q_{76} e^{-\alpha\tilde{t}} + C_7 Q_{77} e^{\beta\tilde{t}} + C_8 Q_{78} e^{-\beta\tilde{t}}, \tag{5.82}$$

$$\psi_{23} = C_5 e^{\alpha\tilde{t}} + C_6 e^{-\alpha\tilde{t}} + C_7 e^{\beta\tilde{t}} + C_8 e^{-\beta\tilde{t}}, \tag{5.83}$$

where Q_{ij} $(1 \le i, j \le 8)$ is the (i, j) comportment of $\boldsymbol{Q}$. In this case, only σ increases with time by the term of $C_4\tilde{t}$. However, this part can be canceled out by the coordinates rotation as $z_I \to z_I e^{\mathrm{i}C_4\tilde{t}}$ and the perturbation in the common angular velocity $\dot{\sigma}$ does not increase with time. Therefore, the Lagrange's solution is stable in the case of $0 < V < 1/27$.

Finally, for the case (iv), we assume $\nu_1 = 1$ and $\nu_2 = \nu_3 = 0$. Then, we obtain

$$\boldsymbol{J} \equiv \begin{pmatrix} \mathrm{i} & 0 & & & & & & \\ & \mathrm{i} & 0 & & & & \mathbf{0} & \\ & & -\mathrm{i} & 0 & & & & \\ & & & -\mathrm{i} & 0 & & & \\ & & & & 0 & 1 & & \\ & & & & & 0 & 0 & \\ & \mathbf{0} & & & & & 0 & 1 \\ & & & & & & & 0 \end{pmatrix}, \tag{5.84}$$

$$\boldsymbol{Q} \equiv \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & \mathrm{i} & 0 & 0 & 0 & 1 \\ \mathrm{i} & 0 & \mathrm{i} & 0 & 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & -\frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{2}{3} \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}. \tag{5.85}$$

Thus, we have

$$\boldsymbol{Y} = (C_1 e^{\mathrm{i}\tilde{t}}, C_2 e^{\mathrm{i}\tilde{t}}, C_3 e^{-\mathrm{i}\tilde{t}}, C_4 e^{-\mathrm{i}\tilde{t}}, C_5 + C_6\tilde{t}, C_6, C_7 + C_8\tilde{t}, C_8), \tag{5.86}$$

where C_i $(i = 1, 2, \cdots, 8)$ is a integral constant. From $\boldsymbol{X} = \boldsymbol{QY}$, we obtain

$$X = \frac{1}{2} C_1 e^{\mathrm{i}(\tilde{t} - \pi/2)} + \frac{1}{2} C_3 e^{-\mathrm{i}(\tilde{t} - \pi/2)} - \frac{2}{3} C_7, \tag{5.87}$$

$$\chi_{12} = \frac{1}{2}C_2 e^{\mathrm{i}(\tilde{t}-\pi/2)} + \frac{1}{2}C_4 e^{-\mathrm{i}(\tilde{t}-\pi/2)} - \frac{2}{3}C_8, \tag{5.88}$$

$$\sigma = C_2 e^{\mathrm{i}\tilde{t}} + C_4 e^{-\mathrm{i}\tilde{t}} + C_6 + C_7 + C_8 + C_8\tilde{t}, \tag{5.89}$$

$$\psi_{23} = C_1 e^{\mathrm{i}\tilde{t}} + C_3 e^{-\mathrm{i}\tilde{t}} + C_5 + C_7 + C_8 + C_7\tilde{t}. \tag{5.90}$$

Equation (5.90) means that the perturbation χ_{23} in the distance r_{23} increases with time, and the Lagrange's solution is unstable. If we assume $\nu_2 = 1$ or $\nu_3 = 1$, we have similar results as χ_{31} or χ_{12} increases with time and the Lagrange's solution is also unstable.

Therefore, the necessary and sufficient condition for the stability of Lagrange's solution is

$$0 < V < \frac{1}{27}. \tag{5.91}$$

This is nothing but the Newtonian condition Eq. (2.52) for stability of Lagrange's solution.

5.2.3 The condition for stability at the 1PN order

Next, let us consider the condition for stability at the 1PN order. [7] The EIH equations of motion (5.50) - (5.53) for the perturbations are

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix} \begin{pmatrix} \chi_{12} \\ \sigma \\ X \\ \psi_{23} \end{pmatrix} = 0, \tag{5.92}$$

where

$$\begin{aligned} M_{11} = (D^2-3) - \frac{1}{32}\varepsilon\Big\{&4\sqrt{3}(\nu_1-\nu_2)(7-9\nu_3)\nu_3 D + (36\nu_2^3 + 234\nu_1\nu_2^2 - 146\nu_2^2 \\ &+ 261\nu_1^2\nu_2 - 488\nu_1\nu_2 + 155\nu_2 + 63\nu_1^3 - 155\nu_1^2 + 137\nu_1 - 585)\Big\}, \end{aligned} \tag{5.93}$$

$$\begin{aligned} M_{12} = -2D - \frac{1}{24}\varepsilon(&27\nu_2^3 + 135\nu_1\nu_2^2 - 21\nu_2^2 + 135\nu_1^2\nu_2 - 210\nu_1\nu_2 + 24\nu_2 + 27\nu_1^3 \\ &- 21\nu_1^2 + 24\nu_1 - 155)D, \end{aligned} \tag{5.94}$$

$$\begin{aligned} M_{13} = -\frac{9}{4}\nu_3 - \frac{1}{32}\varepsilon\nu_3\Big\{&4\sqrt{3}(9\nu_1\nu_2 + 10\nu_2 + 9\nu_1^2 - 6\nu_1 - 4)D - (216\nu_2^2 + 288\nu_1\nu_2 \\ &- 154\nu_2 + 171\nu_1^2 - 38\nu_1 + 420)\Big\}, \end{aligned} \tag{5.95}$$

[7] See also Appendix C.

$$M_{14} = -\frac{3\sqrt{3}}{4}v_3 + \frac{1}{32}\varepsilon v_3\Big\{4(18v_2^2 + 27v_1v_2 - 2v_2 + 9v_1^2 + 14v_1 - 12)D + \sqrt{3}(51v_2^2 + 114v_1v_2 + 2v_2 + 87v_1^2 - 120v_1 + 155)\Big\}, \tag{5.96}$$

$$M_{21} = 2D - \frac{1}{32}\varepsilon\Big\{4(9v_2^3 + 45v_1v_2^2 + 9v_2^2 + 45v_1^2v_2 - 30v_1v_2 - 18v_2 + 9v_1^3 + 9v_1^2 - 18v_1 + 61)D + 3\sqrt{3}v_3(12v_2^2 - 6v_1v_2 + 14v_2 - 15v_1^2 + 4v_1 - 5)\Big\}, \tag{5.97}$$

$$M_{22} = D^2 - \frac{1}{24}\varepsilon\Big\{(3v_2^2 + 12v_1v_2 - 18v_2 + 3v_1^2 - 18v_1 + 10)D^2 - 3\sqrt{3}(v_1 - v_2)v_3 \times (9v_2 + 9v_1 + 4)D\Big\}, \tag{5.98}$$

$$M_{23} = -\frac{3\sqrt{3}}{4}v_3 + \frac{1}{32}\varepsilon v_3\Big\{4(18v_2^2 + 27v_1v_2 + 8v_2 + 9v_1^2 + 16v_1 - 12)D + \sqrt{3}(36v_2^2 + 72v_1v_2 - 54v_2 + 81v_1^2 - 90v_1 + 160)\Big\}, \tag{5.99}$$

$$M_{24} = \frac{9}{4}v_3 + \frac{1}{32}\varepsilon v_3\Big\{4\sqrt{3}(9v_1v_2 + 8v_2 + 9v_1^2 - 4)D - 9(21v_2^2 + 14v_1v_2 - 10v_2 + 13v_1^2 - 8v_1 + 45)\Big\}, \tag{5.100}$$

$$M_{31} = (D^2 - 3) - \frac{1}{32}\varepsilon\Big\{4\sqrt{3}(v_3 - v_1)(7 - 9v_2)v_2D + (36v_3^3 + 234v_1v_3^2 - 146v_3^2 + 261v_1^2v_3 - 488v_1v_3 + 155v_3 + 63v_1^3 - 155v_1^2 + 137v_1 - 585)\Big\}, \tag{5.101}$$

$$M_{32} = -2D - \frac{1}{24}\varepsilon(27v_3^3 + 135v_1v_3^2 - 21v_3^2 + 135v_1^2v_3 - 210v_1v_3 + 24v_3 + 27v_1^3 - 21v_1^2 + 24v_1 - 155)D, \tag{5.102}$$

$$M_{33} = D^2 - 3 + \frac{9}{4}v_2 - \frac{1}{32}\varepsilon\Big\{4\sqrt{3}v_2(9v_3^2 + 9v_1v_3 + 8v_3 - 4v_1 - 4)D - (180v_3^3 + 270v_1v_3^2 - 224v_3^2 + 198v_1^2v_3 + 8v_1v_3 + 419v_3 + 108v_1^3 - 54v_1^2 + 321v_1 + 165)\Big\}, \tag{5.103}$$

$$M_{34} = -\left(2D + \frac{3\sqrt{3}}{4}v_2\right) + \frac{1}{96}\varepsilon\Big\{4(27v_3^3 - 39v_3^2 - 27v_1^2v_3 + 165v_1v_3 - 54v_3 + 36v_1^2 - 102v_1 + 191)D + 3\sqrt{3}v_2(51v_3^2 + 114v_1v_3 + 2v_3 + 87v_1^2 - 120v_1 + 155)\Big\}, \tag{5.104}$$

$$M_{41} = 2D - \frac{1}{32}\varepsilon\Big\{4(9v_3^3 + 45v_1v_3^2 + 9v_3^2 + 45v_1^2v_3 - 30v_1v_3 - 18v_3 + 9v_1^3 + 9v_1^2 - 18v_1 + 61)D - 3\sqrt{3}v_2(12v_3^2 - 6v_1v_3 + 14v_3 - 15v_1^2 + 4v_1 - 5)\Big\}, \tag{5.105}$$

$$M_{42} = D^2 - \frac{1}{24}\varepsilon\Big\{(3\nu_3^2 + 12\nu_1\nu_3 - 18\nu_3 + 3\nu_1^2 - 18\nu_1 + 10)D^2 - 3\sqrt{3}(\nu_3 - \nu_1)$$
$$\times (13 - 9\nu_2)\nu_2 D\Big\}, \tag{5.106}$$
$$M_{43} = 2D - \frac{3\sqrt{3}}{4}\nu_2 + \frac{1}{32}\varepsilon\Big\{4(9\nu_3^3 - 19\nu_3^2 - 9\nu_1^2\nu_3 + 27\nu_1\nu_3 - 2\nu_3 - 2\nu_1^2 - 10\nu_1 - 49)D$$
$$+ \sqrt{3}(72\nu_3^2 + 54\nu_1\nu_3 - 12\nu_3 + 36\nu_1^2 - 78\nu_1 + 145)\nu_2\Big\}, \tag{5.107}$$
$$M_{44} = D^2 - \frac{9}{4}\nu_2 - \frac{1}{96}\varepsilon\Big\{4(3\nu_3^2 + 12\nu_1\nu_3 - 18\nu_3 + 3\nu_1^2 - 18\nu_1 + 10)D^2 - 12\sqrt{3}(9\nu_3^2 + 9\nu_1\nu_3$$
$$+ 12\nu_3 - 4\nu_1 - 4)\nu_2 D - 27(21\nu_3^2 + 14\nu_1\nu_3 - 10\nu_3 + 13\nu_1^2 - 8\nu_1 + 45)\nu_2\Big\}. \tag{5.108}$$

In the same way as the Newtonian case, we obtain the secular equation at the 1PN order as

$$\lambda^2\left[\lambda^6 + 2\left\{1 - \frac{1}{8}\varepsilon(77 - 10V)\right\}\lambda^4 + \left\{1 + \frac{27}{4}V - \frac{1}{16}\varepsilon(308 + 1265V + 162W - 378V^2)\right\}\lambda^2 + \frac{27}{4}\left\{V - \frac{1}{24}\varepsilon(521V - 72W - 126V^2)\right\}\right] = 0, \tag{5.109}$$

where $W \equiv \nu_1\nu_2\nu_3$. [8] If $V = 0$ (so that W also vanishes), we can separate the three-body problem into the problems of two test particles moving around the massive body, and it is not important. Moreover, as we have already shown above, the Lagrange's solution is unstable in the case of $V = 0$ in the Newtonian limit. Hence, we consider $V > 0$ in the following.

Neglecting the trivial root $\lambda = 0$, we obtain a cubic equation of $\tau \equiv \lambda^2$ as

$$\tau^3 + \alpha\tau^2 + \beta\tau + \gamma = 0, \tag{5.110}$$

where

$$\alpha \equiv 2\left\{1 - \frac{1}{8}\varepsilon(77 - 10V)\right\}, \tag{5.111}$$
$$\beta \equiv 1 + \frac{27}{4}V - \frac{1}{16}\varepsilon\left(308 + 1265V + 162W - 378V^2\right), \tag{5.112}$$

[8] From the inequality of arithmetic and geometric means, we have

$$W = \nu_1\nu_2\nu_3 \le \left(\frac{\nu_1 + \nu_2 + \nu_3}{3}\right)^3 = \frac{1}{27},$$

with equality if and only if $\nu_1 = \nu_2 = \nu_3 = 1/3$.

$$\gamma \equiv \frac{27}{4}\left\{V - \frac{1}{24}\varepsilon(521V - 72W - 126V^2)\right\}. \tag{5.113}$$

In the 1PN approximation, the PN corrections to the Newtonian roots must be small. Thus, we can factorize Eq. (5.110) as

$$(\tau + 1 - a\varepsilon)(\tau^2 + b\tau + c) = 0, \tag{5.114}$$

where a, b, and c are constants and the 2PN terms are neglected. From Eqs. (5.110) and (5.114), one can obtain

$$a = \frac{1}{8V}(77V - 14V^2 - 36W), \tag{5.115}$$

$$b = 1 - \frac{1}{8V}(77V - 6V^2 + 36W)\varepsilon, \tag{5.116}$$

$$c = \frac{27}{4}V - \frac{1}{16}(1305V - 378V^2 + 162W)\varepsilon. \tag{5.117}$$

Since $\varepsilon \ll 1$, we have $-1 + a\varepsilon < 0$, $b > 0$, and $c > 0$. The roots of Eq. (5.114) are expressed as

$$\tau_1 = -1 + a\varepsilon, \quad \tau_2 = \frac{-b + \sqrt{b^2 - 4c}}{2}, \quad \tau_3 = \frac{-b - \sqrt{b^2 - 4c}}{2}. \tag{5.118}$$

In a similar manner to the Newtonian case, the PN triangular solution is stable, if and only if τ_2 and τ_3 are negative real numbers and $\tau_2 \neq \tau_3$. Namely, it is necessary and sufficient for the stability that

$$b^2 - 4c > 0. \tag{5.119}$$

The critical value is given by

$$b^2 - 4c = 0. \tag{5.120}$$

Here, V in the PN terms of Eq. (5.120) with Eqs. (5.116) and (5.117) can be replaced by the Newtonian critical value as $V = 1/27$, because its 1PN corrections make 2PN (or higher-order) contributions and they can be neglected. Thus, Eq. (5.120) becomes

$$1-\frac{391}{54}\varepsilon-27\left(V+\frac{15}{2}W\varepsilon\right)=0. \tag{5.121}$$

Therefore, the condition for stability of the PN triangular solution becomes

$$1-\frac{391}{54}\varepsilon-27\left(V+\frac{15}{2}W\varepsilon\right)>0. \tag{5.122}$$

This is explicitly rewritten as

$$\frac{m_1m_2+m_2m_3+m_3m_1}{(m_1+m_2+m_3)^2}+\frac{15}{2}\frac{m_1m_2m_3}{(m_1+m_2+m_3)^3}\varepsilon<\frac{1}{27}\left(1-\frac{391}{54}\varepsilon\right). \tag{5.123}$$

Equation (5.123) recovers the Newtonian condition (2.52) in the limit $\varepsilon \to 0$. The PN correction in the right-hand side of Eq. (5.123) is negative and the PN term of the triple product of masses in the left-hand side of Eq. (5.123) is positive. Hence, the PN condition for stability is tighter than the Newtonian one for any positive values of the parameter ε.

Figure 5.3 shows the Newtonian stability regions given by Eq. (2.52) and the 1PN ones Eq. (5.123) when $\varepsilon = 0.01$ (i.e. the order of magnitude of the PN effects is 0.01), for instance. For values of the mass ratios within the colored areas, the triangular configuration for three finite masses is stable. The stability regions at the 1PN order still exist, and they are more narrow than in the Newtonian case.

Finally, we focus on the restricted three-body limit as $\nu_3 \to 0$ (i.e. $W \to 0$). In this case, the stability condition Eq. (5.119) becomes

$$1-27V-\frac{77-1311V+378V^2}{4}\varepsilon>0. \tag{5.124}$$

This is a quadratic inequality for V. Solving Eq. (5.124) for V, we obtain

$$\frac{m_1m_2}{(m_1+m_2)^2}<\frac{1}{27}\left(1-\frac{391}{54}\varepsilon\right), \tag{5.125}$$

where we used a relation $0 \le V \le 1/3$ in the restricted three-body problem. Using a relation $\nu_1+\nu_2=1$ and assuming $\nu_1>\nu_2$ without loss of generality, we can rewrite Eq. (5.125) as

$$\frac{m_2}{m_1+m_2}<\mu_0-\frac{17\sqrt{69}}{486}\varepsilon, \tag{5.126}$$

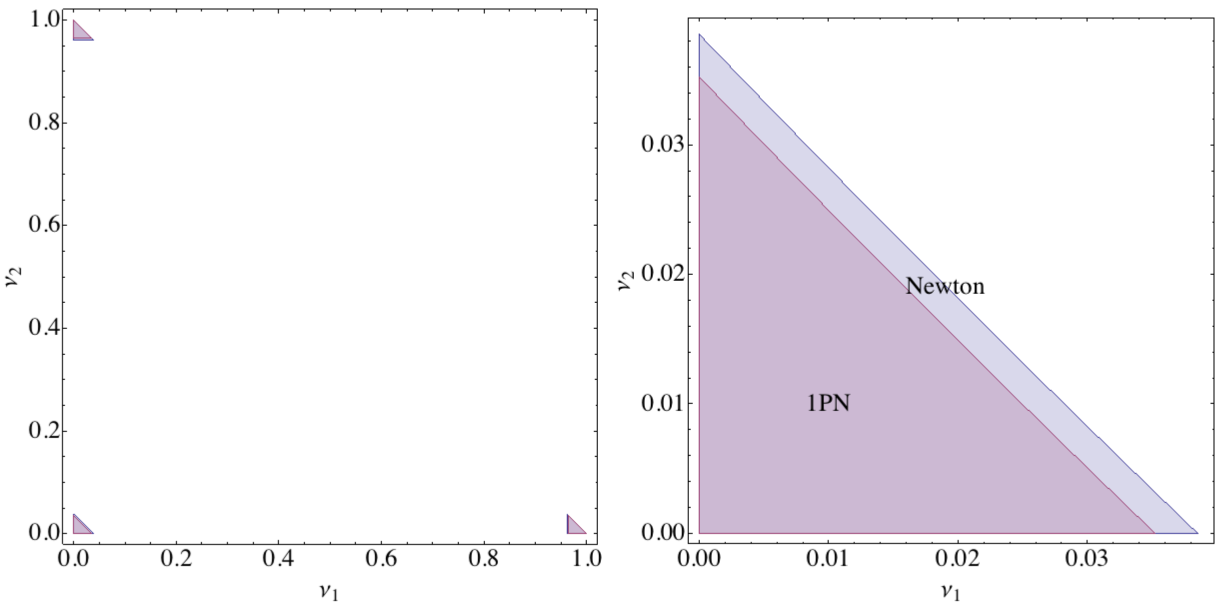

Figure 5.3: The stability regions of the PN triangular solution. Left: All the stability regions. Right: The regions around small ν_1 and ν_2, where the third mass is dominant.

where the Newtonian value $\mu_0 = (9 - \sqrt{69})/18$. This condition is in agreement with the previous results [40, 110, 111].

Figure 5.4 shows a region of stability in the PN restricted three-body problem. For values of ν_2 and ε within the colored area, the PN triangular configuration in the restricted three-body case is stable. This figure corresponds to Fig. 1 in Ref. [40].

The stability regions still exist even at the 1PN order. The PN correction in the right-hand side of Eq. (5.123), which is in agreement with that in the PN restricted three-body problem [40, 110, 111], makes the condition more strict than the Newtonian case for any positive value of the parameter ε.

The PN term of the triple product of masses in the left-hand side of Eq. (5.123) does not appear in the restricted case but in the general one. The instability is also enhanced by this term, while this effect is smaller than the other PN one in the case of mass ratios for stable configurations. If a system is mildly relativistic as $\varepsilon = 0.01$, for instance, the maximum value of W is $O(10^{-4})$ when $\nu_2 = \nu_3 \approx 0.019$ in a stability region. Namely, the contribution from W is only comparable to the 2PN (or more higher) order. This implies that triple systems with the PN triangular configuration for three finite masses are possible as well as restricted three-body systems.

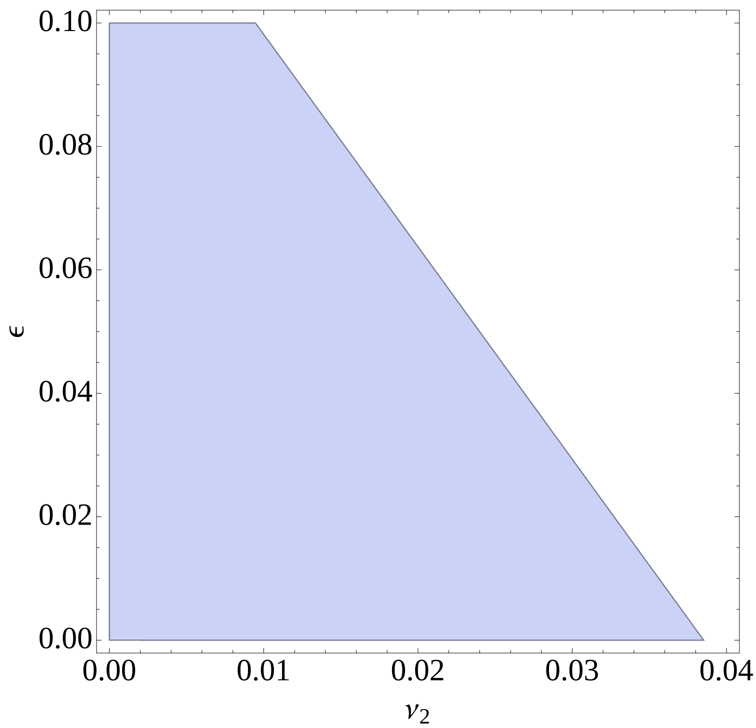

Figure 5.4: The stability region for the restricted PN triangular solution.

The PN triangular configuration ought to emit gravitational waves [9, 119]. Such a system might shrink by gravitational radiation reaction if its configuration is initially stable, and the PN effects on the long-term stability should be incorporated. This is left as a future work. [9]

[9] See also Appendix D.

Chapter 6

Conclusion

We examined the PN effects on equilibrium solutions to the three-body problem for general masses in the framework of general relativity.

First, in the 1PN approximation, we show that a PN collinear configuration corresponding to Euler's one remains an equilibrium solution with the general relativistic corrections to the distances between the bodies. Also, we proved the uniqueness of the collinear configuration for given system parameters (the masses and the end-to-end length). It was shown that a master equation determining the distance ratio among the three bodies, which has been obtained as a septic polynomial, has at most three positive roots, which apparently provide three cases of the distance ratio. It was found, however, that there exists one physically acceptable root and only one. The remaining two positive roots are discarded in the sense that they do not satisfy the slow-motion ansatz in the PN approximation. Since the slow-motion ansatz is a key in the above proof, our way of discussion seems to work at the second (and higher) PN orders, where the slow motion of bodies is assumed. If the configuration has the Newtonian limit, therefore, the uniqueness of collinear configurations for a three-body system may be true even at higher orders. It is an open question whether fully general relativistic systems admit a particular solution that can appear only in a fast motion case and thus has no Newtonian limit. Also, investigating the stability of the PN collinear solution might be of theoretical interest, though the collinear configuration is unstable even in Newtonian gravity.

Next, we showed that an equilateral triangular solution exists at the 1PN order in only two cases: (i) three finite masses are equal and (ii) one mass is finite and the others are zero. Generalizing this, we found a PN triangular equilibrium configuration for general masses with the general relativistic corrections to the distances between the bodies. In addition, we studied the linear stability of the PN triangular configuration and derived the condition for stability at the 1PN order. The stability regions still exist even at the 1PN order. While the PN stability condition recovers the Newtonian one, the PN corrections to the condition always make the condition more strict than in the Newtonian case. The PN term of the triple product of masses does not appear in the restricted case but in the general one. The instability is also enhanced by this term, while this effect is smaller than the other PN one in the case of mass ratios realizing stable configurations. If a system is mildly relativistic as $\varepsilon \sim 10^{-2}$, the maximum value of W is $O(10^{-4})$ corresponding to $\nu_2 = \nu_3 \approx 0.019$ in a stability region. Namely, the contribution from W is only comparable to the 2PN (or more higher) order. This implies that triple systems with the PN triangular configuration for three finite masses are possible as well as restricted three-body systems.

For both these PN solutions, we estimated the magnitude of the PN corrections and we showed that the angular velocity of the bodies are always smaller than in the Newtonian case. These results are useful not only to test general relativity through the high-precision astrometric observations but also to study GWs from the general relativistic three-body systems.

These PN equilibrium configurations ought to emit gravitational waves [9, 119]. It is interesting also to include higher PN corrections, especially 2.5 PN effects in order to elucidate the secular evolution of the orbits due to the gravitational radiation reaction at the 2.5 PN order. One might see probably a shrinking collinear and triangular orbits as a consequence of a decrease in the total energy and angular momentum, if such a radiation reaction effect is included. If such configurations were shrunk by gravitational radiation reaction, the PN effects on the long-term stability should be incorporated. This is left as a future work. Studying the higher order of PN effects and investigating GWs emitted from these solutions are important to test general relativity by GW astronomy in the near future. In order to fully take account of general relativistic effects in the strong field, it might be also interesting to discuss the three- (or more) body problem by the BH perturbation approach as an analytical method.

Appendix A

The PN Center of Mass

In the 1PN approximation, the position defined by [78, 125]

$$\boldsymbol{r}_G \equiv \frac{\sum_A m_A \boldsymbol{r}_A \left[1 + \frac{1}{2}\left\{\left(\frac{v_A}{c}\right)^2 - \sum_{B \neq A} \frac{G m_B}{c^2 r_{AB}}\right\}\right]}{\sum_A m_A \left[1 + \frac{1}{2}\left\{\left(\frac{v_A}{c}\right)^2 - \sum_{B \neq A} \frac{G m_B}{c^2 r_{AB}}\right\}\right]} \tag{A.1}$$

moves with a uniform velocity. We call this, then, the PN center of mass of the system.

We consider an equilateral triangular configuration and put

$$r_{12} = r_{23} = r_{31} = \ell. \tag{A.2}$$

In Eq. (A.1), we can replace v_I and r_{IJ} by the Newtonian values as $v_{\mathrm{N}I}$ and $r_{\mathrm{N}IJ}$, respectively, because their 1PN corrections make 2PN contributions and they can be neglected. Using the Newtonian relations in circular motion, $v_{\mathrm{N}I} = r_{\mathrm{N}I}\omega_{\mathrm{N}}$, $r_{\mathrm{N}1} = \ell\sqrt{\nu_2^2 + \nu_2\nu_3 + \nu_3^2}$, $r_{\mathrm{N}2} = \ell\sqrt{\nu_1^2 + \nu_1\nu_3 + \nu_3^2}$, and $r_{\mathrm{N}3} = \ell\sqrt{\nu_1^2 + \nu_1\nu_2 + \nu_2^2}$, we obtain the expression for the center of mass for the equilateral triangle as

$$\boldsymbol{r}_G = \nu_1\boldsymbol{r}_1 + \nu_2\boldsymbol{r}_2 + \nu_3\boldsymbol{r}_3. \tag{A.3}$$

Therefore, the expression for the PN center of mass reduces to the Newtonian one in the circular equilateral triangle even at the 1PN order.

Appendix B

The PN Corrections to the Angular Velocity of the Collinear Configuration

As we mentioned in Chapter 4, the PN corrections to the common angular velocity of the bodies in the collinear configuration is always negative. This can be proved as follows.

From Eq. (4.29), the dimensionless PN correction is

$$\tilde{\omega}_{\rm PN} \equiv \frac{\ell F_M + F_{\rm N} F_{\rm V}}{2\ell F_{\rm N}}\varepsilon = \frac{GM\varepsilon}{\ell^2 F_{\rm N} z^2 (1+z)^2} \sum_{k=0}^{10} a_k z^k, \tag{B.1}$$

where

$$\begin{aligned}
a_{10} &= -(1-\nu_1-\nu_3)^2 \nu_3^2, \\
a_9 &= -(1-\nu_1-\nu_3)(4+\nu_1-2\nu_3-4\nu_1\nu_3+2\nu_3^2+2\nu_1^2\nu_3-2\nu_1\nu_3^2-4\nu_3^3), \\
a_8 &= -(1-\nu_1-\nu_3)(18+4\nu_1-9\nu_3+3\nu_1^2-14\nu_1\nu_3+2\nu_3^2-\nu_1^3+7\nu_1^2\nu_3+2\nu_1\nu_3^2-6\nu_3^3), \\
a_7 &= -(1-\nu_1-\nu_3)(32+4\nu_1-13\nu_3+12\nu_1^2-18\nu_1\nu_3+10\nu_3^2-4\nu_1^3+8\nu_1^2\nu_3+4\nu_1\nu_3^2-8\nu_3^3), \\
a_6 &= -(30-30\nu_1-37\nu_3+19\nu_1^2-12\nu_1\nu_3+27\nu_3^2-22\nu_1^3+18\nu_1^2\nu_3 \\
&\qquad +12\nu_1\nu_3^2-28\nu_3^3+6\nu_1^4-4\nu_1^3\nu_3-15\nu_1^2\nu_3^2+6\nu_1\nu_3^3+11\nu_3^4), \\
a_5 &= -2(12-13\nu_1-13\nu_3+11\nu_1^2-10\nu_1\nu_3+11\nu_3^2-11\nu_1^3+17\nu_1^2\nu_3 \\
&\qquad +17\nu_1\nu_3^2-11\nu_3^3+4\nu_1^4-3\nu_1^3\nu_3-14\nu_1^2\nu_3^2-3\nu_1\nu_3^3+4\nu_3^4), \\
a_4 &= -(30-37\nu_1-30\nu_3+27\nu_1^2-12\nu_1\nu_3+19\nu_3^2-28\nu_1^3+12\nu_1^2\nu_3 \\
&\qquad +18\nu_1\nu_3^2-22\nu_3^3+11\nu_1^4+6\nu_1^3\nu_3-15\nu_1^2\nu_3^2-4\nu_1\nu_3^3+6\nu_3^4),
\end{aligned}$$

$$
\begin{aligned}
a_3 &= -(1-\nu_1-\nu_3)(32-13\nu_1+4\nu_3+10\nu_1^2-18\nu_1\nu_3+12\nu_3^2-8\nu_1^3+4\nu_1^2\nu_3+8\nu_1\nu_3^2-4\nu_3^3),\\
a_2 &= -(1-\nu_1-\nu_3)(18-9\nu_1+4\nu_3+2\nu_1^2-14\nu_1\nu_3+3\nu_3^2-6\nu_1^3+2\nu_1^2\nu_3+7\nu_1\nu_3^2-\nu_3^3),\\
a_1 &= -(1-\nu_1-\nu_3)(4-2\nu_1+\nu_3+2\nu_1^2-4\nu_1\nu_3-4\nu_1^3-2\nu_1^2\nu_3+2\nu_1\nu_3^2),\\
a_0 &= -(1-\nu_1-\nu_3)^2\nu_1^2.
\end{aligned}
$$

Using a relation $\nu_1+\nu_2+\nu_3=1$, we can prove $a_k<0$ for all the values of $k=0,1,2,\cdots,10$, namely, $\tilde{\omega}_{\rm PN}<0$. Therefore, for the fixed masses m_I and full length a, we have

$$
\omega < \omega_{\rm N}. \tag{B.2}
$$

Appendix C

The Stability of the PN Triangular Solution by Eigenvalue Analysis

Let us derive the condition for stability of the PN triangular solution by eigenvalue analysis. This chapter is based on Ref. [120].

The equations of motion for the perturbations, Eqs. (5.50) - (5.53), can be rewritten as

$$
\begin{aligned}
&(D^2-3)\chi_{12}-2D\sigma-\frac{9}{4}\nu_3 X-\frac{3\sqrt{3}}{4}\nu_3\psi_{23} \\
&\quad+\varepsilon(E_{11}DX+E_{12}X+E_{13}D\chi_{12}+E_{14}\chi_{12}+E_{15}D\sigma+E_{16}\sigma+E_{17}D\psi_{23}+E_{18}\psi_{23})=0, \qquad \text{(C.1)}\\
&2D\chi_{12}+D^2\sigma-\frac{3\sqrt{3}}{4}\nu_3 X+\frac{9}{4}\nu_3\psi_{23} \\
&\quad+\varepsilon(E_{21}DX+E_{22}X+E_{23}D\chi_{12}+E_{24}\chi_{12}+F_1D^2\sigma+E_{25}D\sigma+E_{26}\sigma+E_{27}D\psi_{23} \\
&\qquad+E_{28}\psi_{23})=0, \qquad \text{(C.2)}\\
&(D^2-3)\chi_{12}-2D\sigma+\left(D^2-3+\frac{9}{4}\nu_2\right)X-\left(2D+\frac{3\sqrt{3}}{4}\nu_2\right)\psi_{23} \\
&\quad+\varepsilon(E_{31}DX+E_{32}X+E_{33}D\chi_{12}+E_{34}\chi_{12}+E_{35}D\sigma+E_{36}\sigma+E_{37}D\psi_{23}+E_{38}\psi_{23})=0, \qquad \text{(C.3)}\\
&2D\chi_{12}+D^2\sigma+\left(2D-\frac{3\sqrt{3}}{4}\nu_2\right)X+\left(D^2-\frac{9}{4}\nu_2\right)\psi_{23} \\
&\quad+\varepsilon(E_{41}DX+E_{42}X+E_{43}D\chi_{12}+E_{44}\chi_{12}+F_2D^2\sigma+E_{45}D\sigma+E_{46}\sigma+F_2D^2\psi_{23} \\
&\qquad+E_{47}D\psi_{23}+E_{48}\psi_{23})=0, \qquad \text{(C.4)}
\end{aligned}
$$

where

$$F_1 \equiv -\frac{1}{24}(3v_2^2 + 12v_1v_2 - 18v_2 + 3v_1^2 - 18v_1 + 10), \tag{C.5}$$

$$F_2 \equiv -\frac{1}{24}(3v_3^2 + 12v_1v_3 - 18v_3 + 3v_1^2 - 18v_1 + 10), \tag{C.6}$$

$$E_{11} \equiv -\frac{\sqrt{3}}{8}v_3(9v_1v_2 + 10v_2 + 9v_1^2 - 6v_1 - 4), \tag{C.7}$$

$$E_{12} \equiv \frac{1}{32}v_3(216v_2^2 + 288v_1v_2 - 154v_2 + 171v_1^2 - 38v_1 + 420), \tag{C.8}$$

$$E_{13} \equiv -\frac{\sqrt{3}}{8}(v_1 - v_2)(7 - 9v_3)v_3, \tag{C.9}$$

$$E_{14} \equiv -\frac{1}{32}(36v_2^3 + 234v_1v_2^2 - 146v_2^2 + 261v_1^2v_2 - 488v_1v_2 + 155v_2 + 63v_1^3 - 155v_1^2 + 137v_1 - 585), \tag{C.10}$$

$$E_{15} \equiv -\frac{1}{24}(27v_2^3 + 135v_1v_2^2 - 21v_2^2 + 135v_1^2v_2 - 210v_1v_2 + 24v_2 + 27v_1^3 - 21v_1^2 + 24v_1 - 155), \tag{C.11}$$

$$E_{16} \equiv 0, \tag{C.12}$$

$$E_{17} \equiv \frac{1}{8}v_3(18v_2^2 + 27v_1v_2 - 2v_2 + 9v_1^2 + 14v_1 - 12), \tag{C.13}$$

$$E_{18} \equiv \frac{\sqrt{3}}{32}v_3(51v_2^2 + 114v_1v_2 + 2v_2 + 87v_1^2 - 120v_1 + 155), \tag{C.14}$$

$$E_{21} \equiv \frac{1}{8}v_3(18v_2^2 + 27v_1v_2 + 8v_2 + 9v_1^2 + 16v_1 - 12), \tag{C.15}$$

$$E_{22} \equiv \frac{\sqrt{3}}{32}v_3(36v_2^2 + 72v_1v_2 - 54v_2 + 81v_1^2 - 90v_1 + 160), \tag{C.16}$$

$$E_{23} \equiv -\frac{1}{8}(9v_2^3 + 45v_1v_2^2 + 9v_2^2 + 45v_1^2v_2 - 30v_1v_2 - 18v_2 + 9v_1^3 + 9v_1^2 - 18v_1 + 61), \tag{C.17}$$

$$E_{24} \equiv -\frac{3\sqrt{3}}{32}v_3(12v_2^2 - 6v_1v_2 + 14v_2 - 15v_1^2 + 4v_1 - 5), \tag{C.18}$$

$$E_{25} \equiv \frac{\sqrt{3}}{8}(v_1 - v_2)v_3(9v_1 + 9v_2 + 4), \tag{C.19}$$

$$E_{26} \equiv 0, \tag{C.20}$$

$$E_{27} \equiv \frac{\sqrt{3}}{8}v_3(9v_1v_2 + 8v_2 + 9v_1^2 - 4), \tag{C.21}$$

$$E_{28} \equiv -\frac{9}{32}v_3(21v_2^2 + 14v_1v_2 - 10v_2 + 13v_1^2 - 8v_1 + 45), \tag{C.22}$$

$$E_{31} \equiv -\frac{\sqrt{3}}{8} v_2 (9v_1 v_3 + 8v_3 + 9v_3^2 - 4v_1 - 4), \tag{C.23}$$

$$E_{32} \equiv \frac{1}{32}(180v_3^3 + 270v_1 v_3^2 - 224v_3^2 + 198v_1^2 v_3 + 8v_1 v_3 + 419v_3 + 108v_1^3 - 54v_1^2 + 321v_1 + 165), \tag{C.24}$$

$$E_{33} \equiv -\frac{\sqrt{3}}{8}(v_3 - v_1)(7 - 9v_2)v_2, \tag{C.25}$$

$$E_{34} \equiv -\frac{1}{32}(36v_3^3 + 234v_1 v_3^2 - 146v_3^2 + 261v_1^2 v_3 - 488v_1 v_3 + 155v_3 + 63v_1^3 - 155v_1^2 + 137v_1 - 585), \tag{C.26}$$

$$E_{35} \equiv -\frac{1}{24}(27v_3^3 + 135v_1 v_3^2 - 21v_3^2 + 135v_1^2 v_3 - 210v_1 v_3 + 24v_3 + 27v_1^3 - 21v_1^2 + 24v_1 - 155), \tag{C.27}$$

$$E_{36} \equiv 0, \tag{C.28}$$

$$E_{37} \equiv \frac{1}{24}(27v_3^3 - 39v_3^2 - 27v_1^2 v_3 + 165v_1 v_3 - 54v_3 + 36v_1^2 - 102v_1 + 191), \tag{C.29}$$

$$E_{38} \equiv \frac{\sqrt{3}}{32} v_2 (51v_3^2 + 114v_1 v_3 + 2v_3 + 87v_1^2 - 120v_1 + 155), \tag{C.30}$$

$$E_{41} \equiv \frac{1}{8}(9v_3^3 - 19v_3^2 - 9v_1^2 v_3 + 27v_1 v_3 - 2v_3 - 2v_1^2 - 10v_1 - 49), \tag{C.31}$$

$$E_{42} \equiv \frac{\sqrt{3}}{32} v_2 (72v_3^2 + 54v_1 v_3 - 12v_3 + 36v_1^2 - 78v_1 + 145), \tag{C.32}$$

$$E_{43} \equiv -\frac{1}{8}(9v_3^3 + 45v_1 v_3^2 + 9v_3^2 + 45v_1^2 v_3 - 30v_1 v_3 - 18v_3 + 9v_1^3 + 9v_1^2 - 18v_1 + 61), \tag{C.33}$$

$$E_{44} \equiv \frac{3\sqrt{3}}{32} v_2 (12v_3^2 - 6v_1 v_3 + 14v_3 - 15v_1^2 + 4v_1 - 5), \tag{C.34}$$

$$E_{45} \equiv \frac{\sqrt{3}}{8}(v_3 - v_1)v_2(13 - 9v_2), \tag{C.35}$$

$$E_{46} \equiv 0, \tag{C.36}$$

$$E_{47} \equiv \frac{\sqrt{3}}{8} v_2 (9v_3^2 + 9v_1 v_3 + 12v_3 - 4v_1 - 4), \tag{C.37}$$

$$E_{48} \equiv \frac{9}{32} v_2 (21v_3^2 + 14v_1 v_3 - 10v_3 + 13v_1^2 - 8v_1 + 45). \tag{C.38}$$

Defining new variables $\dot{X} \equiv DX$, $\dot{\chi}_{12} \equiv D\chi_{12}$, $\dot{\sigma} \equiv D\sigma$, and $\dot{\psi}_{23} \equiv D\psi_{23}$, we obtain from Eqs. (C.1)

- (C.4)

$$
\begin{aligned}
D\dot{\chi}_{12} &= -\varepsilon E_{11}\dot{X} - \varepsilon E_{13}\dot{\chi}_{12} + (2-\varepsilon E_{15})\dot{\sigma} - \varepsilon E_{17}\dot{\psi}_{23} \\
&\quad + \left(\frac{9}{4}\nu_3 - \varepsilon E_{12}\right)X + (3-\varepsilon E_{14})\chi_{12} + \left(\frac{3\sqrt{3}}{4}\nu_3 - \varepsilon E_{18}\right)\psi_{23}, \qquad \text{(C.39)}\\
D\dot{\sigma} &= \Big[-\varepsilon E_{21}\dot{X} - (2+\varepsilon E_{23})\dot{\chi}_{12} - \varepsilon E_{25}\dot{\sigma} - \varepsilon E_{27}\dot{\psi}_{23} \\
&\quad + \left(\frac{3\sqrt{3}}{4}\nu_3 - \varepsilon E_{22}\right)X - \varepsilon E_{24}\chi_{12} - \left(\frac{9}{4}\nu_3 + \varepsilon E_{28}\right)\psi_{23}\Big]\kappa_1, \qquad \text{(C.40)}\\
D\dot{X} &= -\varepsilon E_{51}\dot{X} - \varepsilon E_{53}\dot{\chi}_{12} - \varepsilon E_{55}\dot{\sigma} + (2-\varepsilon E_{57})\dot{\psi}_{23} \\
&\quad + \left(3 - \frac{9}{4}(\nu_2+\nu_3) - \varepsilon E_{52}\right)X - \varepsilon E_{54}\chi_{12} + \left(\frac{3\sqrt{3}}{4}(\nu_2-\nu_3) - \varepsilon E_{58}\right)\psi_{23}, \qquad \text{(C.41)}\\
D\dot{\psi}_{23} &= -(2\kappa_2 + \varepsilon E_{61})\dot{X} - (2\kappa_2 - 2\kappa_1 + \varepsilon E_{63})\dot{\chi}_{12} - \varepsilon E_{65}\dot{\sigma} - \varepsilon E_{67}\dot{\psi}_{23} \\
&\quad + \left\{\frac{3\sqrt{3}}{4}(\kappa_2\nu_2 - \kappa_1\nu_3) - \varepsilon E_{62}\right\}X - \varepsilon E_{64}\chi_{12} + \left\{\frac{9}{4}(\kappa_2\nu_2 + \kappa_1\nu_3) - \varepsilon E_{68}\right\}\psi_{23}, \qquad \text{(C.42)}
\end{aligned}
$$

where $E_{5i} \equiv E_{3i} - E_{1i}$ $(i = 1,\cdots,8)$, $E_{6i} \equiv \kappa_2 E_{4i} - \kappa_1 E_{2i}$ $(i = 1,\cdots,8)$, and $\kappa_i \equiv \dfrac{1}{1+\varepsilon F_i} = 1 - \varepsilon F_i + \mathrm{O}(\varepsilon^2)$ $(i = 1,2)$.

Therefore, we have

$$D\boldsymbol{X} = \mathscr{M}\boldsymbol{X}, \tag{C.43}$$

$$\mathscr{M} \equiv \mathscr{M}_N - \varepsilon \begin{pmatrix} E_{51} & E_{53} & E_{55} & E_{57} & E_{52} & E_{54} & 0 & E_{58} \\ E_{11} & E_{13} & E_{15} & E_{17} & E_{12} & E_{14} & 0 & E_{18} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ E_{61} & E_{63} & E_{65} & E_{67} & E_{62} & E_{64} & 0 & E_{68} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+\kappa_1 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\varepsilon E_{21} & -2-\varepsilon E_{23} & -\varepsilon E_{25} & -\varepsilon E_{27} & \frac{3\sqrt{3}}{4}\nu_3 - \varepsilon E_{22} & -\varepsilon E_{24} & 0 & -\frac{9}{4}\nu_3 - \varepsilon E_{28} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$-\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & \frac{3\sqrt{3}}{4}\nu_3 & 0 & 0 & -\frac{9}{4}\nu_3 \\ -2+2\kappa_2 & 2(\kappa_2-\kappa_1) & 0 & 0 & \frac{3\sqrt{3}}{4}(\nu_2-\nu_3-\kappa_2\nu_2+\kappa_1\nu_3) & 0 & 0 & \frac{9}{4}(\nu_2+\nu_3-\kappa_2\nu_2-\kappa_1\nu_3) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{C.44}$$

where $\boldsymbol{X} \equiv (\dot{X}, \dot{\chi_{12}}, \dot{\sigma}, \dot{\psi_{23}}, X, \chi_{12}, \sigma, \psi_{23})$ and we define the Newtonian matrix

$$\mathscr{M}_N \equiv \begin{pmatrix} 0 & 0 & 0 & 2 & 3\left(1-\frac{3}{4}(\nu_2+\nu_3)\right) & 0 & 0 & \frac{3\sqrt{3}}{4}(\nu_2-\nu_3) \\ 0 & 0 & 2 & 0 & \frac{9}{4}\nu_3 & 3 & 0 & \frac{3\sqrt{3}}{4}\nu_3 \\ 0 & -2 & 0 & 0 & \frac{3\sqrt{3}}{4}\nu_3 & 0 & 0 & -\frac{9}{4}\nu_3 \\ -2 & 0 & 0 & 0 & \frac{3\sqrt{3}}{4}(\nu_2-\nu_3) & 0 & 0 & \frac{9}{4}(\nu_2+\nu_3) \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{C.45}$$

The eigenvalue equation for the coefficient matrix $\mathscr{M}$ is

$$\lambda^2\Big\{\lambda^6+\frac{1}{4}\lambda^4(8-\varepsilon(77-10V))+\frac{1}{16}\lambda^2\{4(4+27V)+\varepsilon(378V^2-1265V-162W-308)\} \\ +\frac{9}{32}\{24V+\varepsilon(126V^2-521V+72W)\}\Big\}=0, \tag{C.46}$$

or

$$\tau(\tau^3+A\tau^2+B\tau+C)=0, \tag{C.47}$$

where λ is the eigenvalue, $V=\nu_1\nu_2+\nu_2\nu_3+\nu_3\nu_1$, $W=\nu_1\nu_2\nu_3$, and $\tau\equiv\lambda^2$. The coefficients in Eq. (C.47) are defined by

$$A \equiv \frac{1}{4}(8-\varepsilon(77-10V)), \tag{C.48}$$

$$B \equiv \frac{1}{16}(4(4+27V)+\varepsilon(378V^2-1265V-162W-308)), \tag{C.49}$$

$$C \equiv \frac{9}{32}(24V+\varepsilon(126V^2-521V+72W)). \tag{C.50}$$

Since $\varepsilon \ll 1$, we have $A>0$, $B>0$, and $C>0$. Therefore, from Eq. (C.47), we obtain two eigenvalues as the multiple roots $\lambda_0=0$ and the six other eigenvalues λ_i $(i=1,\cdots,6)$. We define a cubic function as

$$f(\tau) \equiv \tau^3 + A\tau^2 + B\tau + C. \tag{C.51}$$

For Eq. (C.43), we consider the Jordan normal form of $\mathscr{M}$ as $J \equiv Q^{-1}\mathscr{M}Q$, where Q is the transition matrix. Defining the new vector as $\boldsymbol{Y} \equiv Q^{-1}\boldsymbol{X}$, Eq. (C.43) can be rewritten as

$$\boldsymbol{DY} = \boldsymbol{JY}, \tag{C.52}$$

$$\boldsymbol{J} \equiv \begin{pmatrix} 0 & 1 & & & & & & \\ & 0 & 0 & & & & \mathbf{0} & \\ & & \lambda_1 & s_1 & & & & \\ & & & \lambda_2 & s_2 & & & \\ & & & & \lambda_3 & s_3 & & \\ & & & & & \lambda_4 & s_4 & \\ & & \mathbf{0} & & & & \lambda_5 & s_5 \\ & & & & & & & \lambda_6 \end{pmatrix}, \tag{C.53}$$

where if $\lambda_i = \lambda_{i+1}$, then $s_i = 1$, else $s_i = 0$.

For real numbers a and b, we consider the following three cases:

- if $\tau_i = a > 0$, then $\lambda_i = \pm\sqrt{a}$, so that $\boldsymbol{Y}$ contains the terms proportional to $e^{\pm\sqrt{a}t}$.
- If $\tau_i = a < 0$, then $\lambda_i = \pm\mathrm{i}\sqrt{a}$, so that $\boldsymbol{Y}$ contains the terms proportional to $e^{\pm\mathrm{i}\sqrt{a}t}$.
- If $\tau_i = a + \mathrm{i}b$, then $\lambda_i = \pm(p + \mathrm{i}q)$, where $p = \dfrac{b}{2}\sqrt{\dfrac{2}{-a+\sqrt{a^2+b^2}}}$ and $q = \sqrt{\dfrac{-a+\sqrt{a^2+b^2}}{2}}$, so that $\boldsymbol{Y}$ contains the terms proportional to $e^{\pm(p+\mathrm{i}q)t}$.

The PN triangular solution is stable, if and only if $\tau_i < 0$ for all the values of $i = 1, 2, \cdots, 6$, so that the perturbations $\boldsymbol{X}$ consists of the linear combination of $e^{\pm\mathrm{i}\sqrt{a}t}$. This is because that the perturbations diverge with time in the other cases.

In the Newtonian limit, $f(\tau) = 0$ has no multiple root. Since the PN corrections to the Newtonian roots are sufficiently small, we consider three roots with the PN corrections, where the roots are different from each other. [1] $f(\tau) = 0$ has three real negative roots, if and only if the following

[1] If some of the three roots are equal to each other, that is, multiple roots, the perturbations $\boldsymbol{X}$ contain unstable modes.

three conditions (I), (II), and (III) hold. (I) $C > 0$. (II) $df(\tau)/d\tau = 0$ has two negative roots. (III) All the three roots to $f(\tau) = 0$ are different from each other.

The condition (I) becomes

$$24V + \varepsilon(126V^2 - 521V + 72W) > 0. \tag{C.54}$$

The condition (II) holds, if and only if $A^2 - 3B > 0$ and $-A + \sqrt{A^2 - 3B} < 0$ and $-A - \sqrt{A^2 - 3B} < 0$, because the roots of $df(\tau)/d\tau = 0$ are

$$\tau = \frac{-A \pm \sqrt{A^2 - 3B}}{3}. \tag{C.55}$$

The condition (III) is equivalent to the condition as

$$D \equiv \frac{1}{27}(-A^2B^2 + 4B^3 + 4A^3C - 18ABC + 27C^2) < 0. \tag{C.56}$$

D is the discriminant.

Here, if $A^2 - 3B \leq 0$, then

$$D = \left\{C + \frac{1}{27}(2A^2 - 9B)A\right\}^2 + \left(\frac{2}{27}\sqrt{(A^2 - 3B)^3}\right)^2 \geq 0. \tag{C.57}$$

Therefore, if $D < 0$, then $A^2 - 3B > 0$. In addition, in this case, since $A > 0$, $B > 0$, and $C > 0$, the conditions (I) and (II) automatically hold. Hence, the condition (C.56) is the necessary and sufficient condition for stability of the PN triangle.

It seems that the condition Eq. (C.56) is not equivalent to the condition Eq. (5.119). However, this is not the case. In fact, for $\varepsilon \ll 1$, we can show

$$D = h(b^2 - 4c), \tag{C.58}$$

where

$$h \equiv \frac{729V^2}{16} + \frac{27}{32}(-1289V^2 + 378V^3 + 144W - 162VW)\varepsilon > 0. \tag{C.59}$$

Therefore, the condition Eq. (C.56) and the condition Eq. (5.119) are equivalent to each other in the 1PN approximation.

Appendix D

A Preliminary Note of Gravitational Radiation Reactions on Lagrange's Solution

D.1 The reaction force by gravitational radiation

The *reduced quadrupole moment* of the mass distribution is [84, 81]

$$I_{jk} = \sum_J G m_J \left(x_j^J x_k^J - \frac{1}{3} \delta_{jk} r_J^2 \right), \tag{D.1}$$

where m_J, x_j^J, and r_J are the mass, position, and distance from the center of mass of the Jth body.

In the Newtonian limit, we assume that all of the bodies move on circular orbits in the same plane. Using a complex plane as the orbital one, we denote the positions of the bodies as

$$z_I = r_I e^{i\theta_I}, \quad \theta_I = \omega t + \varphi_I, \tag{D.2}$$

where φ_I is the initial direction of the Ith body, and we assume that each body moves with the common angular velocity ω. We adopt the x and y axes to the real and imaginary ones, and hence, we have

$$\bar{I}_{xx} = \sum_J G m_J r_J^2 \cos^2\theta_J + \text{constant}, \tag{D.3}$$

$$\bar{I}_{xy} = \bar{I}_{yx} = \sum_J G m_J r_J^2 \cos\theta_J \sin\theta_J, \tag{D.4}$$

$$\bar{I}_{yy} = \sum_J G m_J r_J^2 \sin^2\theta_J + \text{constant}, \tag{D.5}$$

$$\bar{I}_{zz} = \text{constant}, \tag{D.6}$$

and the other components vanish. Therefore, their fifth derivatives with respect to time are

$$\frac{d^5\bar{I}_{xx}}{d(ct)^5} = -16\frac{G\omega^5}{c^5}\sum_J m_J r_J^2 \sin(2\theta_J), \tag{D.7}$$

$$\frac{d^5\bar{I}_{xy}}{d(ct)^5} = 16\frac{G\omega^5}{c^5}\sum_J m_J r_J^2 \cos(2\theta_J) = \frac{d^5\bar{I}_{yx}}{d(ct)^5}, \tag{D.8}$$

$$\frac{d^5\bar{I}_{yy}}{d(ct)^5} = 16\frac{G\omega^5}{c^5}\sum_J m_J r_J^2 \sin(2\theta_J) = -\frac{d^5\bar{I}_{xx}}{d(ct)^5}. \tag{D.9}$$

The gravitational acceleration to a particle at point $\boldsymbol{r}$ by gravitational radiation is expressed by [84, 81]

$$\boldsymbol{F}^{\text{GW}} = -\frac{\partial\Phi^{\text{GW}}}{\partial\boldsymbol{r}}, \tag{D.10}$$

where we denote

$$\Phi^{\text{GW}} = \frac{1}{5}\frac{d^5\bar{I}_{jk}}{d(ct)^5}x^j x^k. \tag{D.11}$$

Therefore,

$$\left[\boldsymbol{F}^{\text{GW}}\right]_j = -\frac{2}{5}\frac{d^5\bar{I}_{jk}}{d(ct)^5}x^k, \tag{D.12}$$

and then we obtain

$$\left[\boldsymbol{F}^{\mathrm{GW}}\right]_z \propto \frac{d^5 I_{zk}}{d(ct)^5} x^k = \frac{d^5 I_{zz}}{d(ct)^5} x^z = 0. \tag{D.13}$$

This means that the bodies move in the (x,y) plane even if we take account of the radiation reaction. We discuss the motion of the bodies in the orbital plane in the following.

Since the bodies moves on the circular orbit, we can denote the position as $\boldsymbol{r} = r(\cos\theta, \sin\theta)$. Hence, the radiation reaction force at $\boldsymbol{r}$ becomes

$$\begin{aligned}
\boldsymbol{F}^{\mathrm{GW}} &= \frac{32}{5}\frac{Gr\omega^5}{c^5}\sum_J m_J r_J^2 \begin{bmatrix} \sin(2\theta_J) & -\cos(2\theta_J) \\ -\cos(2\theta_J) & -\sin(2\theta_J) \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \\
&= -\frac{32}{5}\frac{GM}{\ell^2}\varepsilon\bar{r}\sum_J \nu_J \bar{r}_J^2 \begin{bmatrix} \sin(2\theta - 2\theta_J) & -\cos(2\theta - 2\theta_J) \\ \cos(2\theta - 2\theta_J) & \sin(2\theta - 2\theta_J) \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix},
\end{aligned} \tag{D.14}$$

where we denoted a dimensionless radius as $\bar{r}_I \equiv \frac{r_I}{\ell}$ for a constant ℓ of the dimension of length and we defined the small parameter of the order of 2.5PN as

$$\varepsilon \equiv \left(\frac{\ell\omega}{c}\right)^5. \tag{D.15}$$

. This can be also expressed as

$$F^{\mathrm{GW}} = -\frac{32}{5}\frac{GM}{\ell^2}\varepsilon\bar{r}\sum_J \nu_J \bar{r}_J^2 [\sin(2\theta - 2\theta_J) + i\cos(2\theta - 2\theta_J)] e^{i\theta}. \tag{D.16}$$

D.2 Applications to Lagrange's solution

We apply the above formulations to Lagrange's solution (see Subsection 2.3.2). The position of each body is

$$z_I = r_I e^{i\theta_I}, \tag{D.17}$$

where r_I and θ_I are the radius and the angle from a reference direction, respectively. We denote the relative position of the bodies is

$$z_{IJ} = z_I - z_J = r_{IJ} e^{i\theta_{IJ}}, \tag{D.18}$$

where r_{IJ} and θ_{IJ} are the distance between the bodies and the angle from the reference direction, respectively.

We choose the center of mass as the origin of the coordinates:

$$\nu_1 z_1 + \nu_2 z_2 + \nu_3 z_3 = 0. \tag{D.19}$$

In this case, the radiation reaction (D.16) to the Ith body becomes

$$\begin{aligned} F_I^{\mathrm{GW}} &= -\frac{32}{5}\frac{GM}{\ell^2}\varepsilon r_I \sum_J \nu_J r_J^2 [\sin(2\varphi_{IJ}) + i\cos(2\varphi_{IJ})] e^{i\theta_I} \\ &= -\frac{32}{5}\frac{GM}{\ell^2}\varepsilon r_I [A_I + iB_I] e^{i\theta_I}, \end{aligned} \tag{D.20}$$

where we have denoted $\varphi_{IJ} \equiv \theta_I - \theta_J = \varphi_I - \varphi_J$ and omit bars over the letter r for simplicity. A_I and B_I are defined as

$$A_I \equiv \sum_J \nu_J r_J^2 \sin(2\varphi_{IJ}), \quad B_I \equiv \sum_J \nu_J r_J^2 \cos(2\varphi_{IJ}), \tag{D.21}$$

respectively. In Newtonian gravity, Lagrange's configuration is an equilibrium one, so that r_J and φ_{IJ} are the constants. Then, A_I and B_I are also constants at the first order of ε.

In order to consider only the effects of the dominant part of the radiation reaction, we add the reaction force to the Newtonian gravity and neglect other PN terms. Then, the equation of motion for the Ith body is

$$\begin{aligned} \left[(\ddot{r}_I - r_I\dot{\theta}_I^2) + i(2\dot{r}_I\dot{\theta}_I + r_I\ddot{\theta}_I)\right] e^{i\theta_I} &= -\frac{GM}{\ell^3}\frac{r_I}{R^3} e^{i\theta_I} + F_I^{GW} \\ &= -\frac{GM}{(\ell R)^3} r_I \left[1 + \frac{32}{5}R^3 (A_I + iB_I)\varepsilon\right] e^{i\theta_I}, \end{aligned} \tag{D.22}$$

where R is each side length normalized by ℓ.

In the Newtonian limit (i.e. $\varepsilon \to 0$), this equation admits an equilateral triangular solution:

$$r_I = r_I^{\mathrm{N}} = \sqrt{\nu_J^2 + \nu_J\nu_K + \nu_K^2}, \quad R = R^{\mathrm{N}}, \quad \dot{\theta}_I = \omega^{\mathrm{N}} = \sqrt{\frac{GM}{(\ell R^{\mathrm{N}})^3}}. \tag{D.23}$$

We assume the solution to Eq. (D.22) as

$$r_I = r_I^{\mathrm{N}} + r_I^{\mathrm{PN}}\varepsilon, \quad R = R^{\mathrm{N}} + R^{\mathrm{PN}}\varepsilon, \quad \dot{\theta}_I = \omega^{\mathrm{N}} + \omega_I^{\mathrm{PN}}\varepsilon, \tag{D.24}$$

where $r_I^{\rm PN}$, $R^{\rm PN}$, and $\omega^{\rm PN}$ are the 2.5PN corrections and they are functions of time. We assume the bodies move with the same angular velocity

$$\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \omega, \tag{D.25}$$

namely,

$$\omega_1^{\rm PN} = \omega_2^{\rm PN} = \omega_3^{\rm PN} = \omega^{\rm PN}. \tag{D.26}$$

Also, we assume that the equilateral triangle is similarly deformed by the radiation reaction. For this, it is necessary that

$$\frac{r_1^{\rm PN}}{r_1^{\rm N}} = \frac{r_2^{\rm PN}}{r_2^{\rm N}} = \frac{r_3^{\rm PN}}{r_3^{\rm N}} = \frac{R^{\rm PN}}{R^{\rm N}} \equiv \chi. \tag{D.27}$$

Therefore, the real and imaginary parts of the equation of motion (D.22) become

$$\ddot{\chi} - 3(\omega^{\rm N})^2\chi - 2\omega^{\rm N}\omega^{\rm PN} = -\frac{32}{5}(\omega^{\rm N})^2(R^{\rm N})^3 A_I, \tag{D.28}$$

$$2\dot{\chi}\omega^{\rm N} + \dot{\omega}^{\rm PN} = -\frac{32}{5}(\omega^{\rm N})^2(R^{\rm N})^3 B_I, \tag{D.29}$$

respectively, where we neglected the higher order, and then the left-hand sides in Eqs. (D.28) and (D.29) are constants.

We consider that the solutions to Eqs. (D.28) and (D.29) are in agreement with the Newtonian ones at initial time, namely, $\chi(t=0) = \omega^{\rm PN}(t=0) = 0$. As a result, we obtain the solutions as

$$\omega^{\rm PN} = \frac{32}{5}(R^{\rm N})^3\omega^{\rm N}\left[2A_I + 3B_I\omega^{\rm N}t - (A_I - 2iB_I)e^{i\omega^{\rm N}t} - (A_I + 2iB_I)e^{-i\omega^{\rm N}t}\right], \tag{D.30}$$

$$\chi = -\frac{16}{5}(R^{\rm N})^3\left[2A_I + 4B_I\omega^{\rm N}t - (A_I - 2iB_I)e^{i\omega^{\rm N}t} - (A_I + 2iB_I)e^{-i\omega^{\rm N}t}\right]. \tag{D.31}$$

The solutions (D.30) and (D.31) diverge in the limit as $t \to \infty$. This means that the PN approximation will not be valid after a long time. Therefore, it is necessary that $\omega^{\rm PN}$ and χ expressed as Eqs. (D.30) and (D.31) are of the order of unity, that is, $\omega^{\rm N}t \sim 1$. Also, the assumptions Eqs. (D.26) and (D.27) are valid, in the case that

$$A_1 = A_2 = A_3 \tag{D.32}$$

and

$$B_1 = B_2 = B_3 \tag{D.33}$$

hold simultaneously. The conditions Eqs. (D.32) and (D.33) hold if and only if

$$m_1 = m_2 = m_3. \tag{D.34}$$

In this case,

$$A_1 = A_2 = A_3 = B_1 = B_2 = B_3 = 0, \tag{D.35}$$

so that no gravitational wave is emitted.

In the restricted three-body case (i.e. $m_3 \to 0$), the orbits of the binary with the massive two bodies are inspirals, while the equilateral triangle breaks down. Hence, by comparing with the case of the circular binary system, we might check whether the above calculations are correct. In this case, from Eq. (D.21), we obtain

$$A_1 = A_2 = 0, \tag{D.36}$$

$$B_1 = B_2 = \eta, \tag{D.37}$$

where $\eta \equiv \nu_1 \nu_2$. Then, Eqs. (D.30) and (D.31) become

$$\omega^{\mathrm{PN}} = \frac{32}{5}\omega^{\mathrm{N}}\eta\left[3\omega^{\mathrm{N}}t - 4\sin(\omega^{\mathrm{N}}t)\right], \tag{D.38}$$

$$\chi = -\frac{64}{5}\eta\left[\omega^{\mathrm{N}}t - \sin(\omega^{\mathrm{N}}t)\right]. \tag{D.39}$$

Time variation of the angular velocity is

$$\dot{\omega} = \dot{\omega}^{\mathrm{PN}}\varepsilon + \mathrm{O}(\varepsilon^2) = \frac{96}{5}(\omega^{\mathrm{N}})^2\eta\left[1 - \frac{4}{3}\cos(\omega^{\mathrm{N}}t)\right]\varepsilon + \mathrm{O}(\varepsilon^2). \tag{D.40}$$

Averaging this over one Newtonian orbital period $T^{\mathrm{N}} = 2\pi/\omega^{\mathrm{N}}$, we have

$$\frac{1}{T^{\mathrm{N}}}\int_0^{T^{\mathrm{N}}} dt\dot{\omega}^{\mathrm{PN}}\varepsilon = \frac{96}{5}(\omega^{\mathrm{N}})^2\eta\varepsilon. \tag{D.41}$$

Therefore, the (averaged) change rate of the orbital period is given by

$$\frac{\dot{T}}{T} = -\frac{96}{5}\frac{G^3 M^3}{\ell^4 c^5}\eta + \mathrm{O}(\varepsilon^2). \tag{D.42}$$

This is in agreement with the case of the circular binary system [84, 81].

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