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Some Lemmas related to the Blowup Problem of Compressible Burgers Equations

圧縮性バーガース方程式の爆発問題に関連したいくつかの補題

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Abstract : In this paper, we have derived a priori estimates which are required to discuss the temporal behavior of the spatially spherosymmetric solution to the 3-dimensional compressible Burgers equation.

Key words : compressible Burgers equations , blowup problem

1. Presentation of the problem

The final settlement of the time-global problem of the solution for the compressible Navier-Stokes equations,

$$(1.1)\qquad \begin{cases} \rho_t + \mathrm{div}(\rho v) = 0 \\ \rho\{v_t + (v\cdot\nabla)v\} = \mu\left(\Delta + \frac{1}{3}\nabla\mathrm{div}\right)v - \nabla R\rho\theta \\ c_V\rho\{\theta_t + (v\cdot\nabla)\theta\} = \kappa\Delta\theta - R\rho\theta\,\mathrm{div}\,v - \Psi[\nabla v] \end{cases}$$

as yet, seems to be too far for us to attain. Here $\rho = \rho(x,t)$ is the density, $v = v(x,t)$ is the velocity vector, $\theta = \theta(x,t)$ is the absolute temperature, μ is the viscosity coefficient, κ is the heat conductivity, c_V is the specific heat at constant volume, R is the gas constant, $\Psi[\nabla v] = -\frac{2}{3}(\mathrm{div}\,v)^2 + \frac{\mu}{2}\sum_{i,j=1}^{3}\left(\frac{\partial}{\partial x_i}v_j + \frac{\partial}{\partial x_j}v_i\right)^2$ is the constipatim function. In addition, μ, κ and c_V are positive constants.

In this situation, here we try to obtain some results concerning the blow up of the solution for a simplified model system of (1.1) which has the following form,

$$(1.2)\qquad \begin{cases} \rho_t + \mathrm{div}(\rho v) = 0 \\ \rho\{v_t + (v\cdot\nabla)v\} = \mu\left(\Delta + \frac{1}{3}\nabla\mathrm{div}\right)v \end{cases}$$

We call the system of equations (1.2) ” 3-dimensional compressible Burgers equations”.
Hereafter, we consider the initial-boundary value problem for (1.2) in $O_{\ell,T} = O_\ell \times [0,T]$, where $O_\ell = \{x \in \mathbb{R}^3 \,|\, |x| \le \ell\}$ and $0 < T \le \infty$ with given conditions,

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$$
(1.2)' \qquad \begin{cases} v(x,0) = v_0(x) = \begin{cases} 0 \; (|x|=0) \\ \tilde{v}_0(|x|)\dfrac{x}{|x|} \; (|x| \neq 0) \end{cases} \\ \tilde{v}_0(r) \in H^{2+\alpha}(I_\ell) \; (I_\ell = [0,\ell], \; 0<\alpha<1), \;\; \tilde{v}_0(0) = \tilde{v}_0(\ell) = 0 \\ v(x,t)|_{|x|=\ell} = 0 \; (t \geq 0), \;\; \rho(x,0) = \rho_0(x) = \tilde{\rho}_0(|x|) = \overline{\rho}_0 > 0 \\ \tilde{v}_0''(\ell) + \dfrac{2}{\ell}\tilde{v}_0'(\ell) = 0 \end{cases}
$$

The notations used are conventional (e.g. [1]), so that we do not make any particular comment on them, unless necessity arises.

Without proof (see [1] and [2]), we state

Theorem 1.1. *There exists a unique solution* (v, ρ) *of* $(1.2)-(1.2)'$ *which belongs to* $\left(H^{2+\alpha,1+\alpha/2}(O_{\ell,T})\right)^3 \times B^{1+\alpha,1+\alpha/2}(O_{\ell,T})$.
Moreover, noting the properties of $v_0(x)$ *and* $\rho_0(x)$, v *and* ρ *have form*

$$
(1.3) \qquad \begin{cases} v(x,t) = \begin{cases} 0 \;\; (|x|=0) \\ \tilde{v}(|x|,t)\dfrac{x}{|x|} \;\; (|x| \neq 0) \end{cases} \\ \rho(x,t) = \tilde{\rho}(|x|,t) \end{cases}
$$

where $\tilde{v}(r,t)$ *and* $\tilde{\rho}(r,t)$ $(r=|x|<\ell)$ *satisfy*

$$
(1.3)' \qquad \begin{cases} \tilde{\rho}(r,t)\left\{\tilde{v}_t(r,t) + \dfrac{2}{r}\tilde{v}\tilde{v}_r\right\} = \overline{\mu}\left(\tilde{v}_{rr} + \dfrac{2}{r}\tilde{v}_r - \dfrac{2}{r^2}\tilde{v}\right), \; \overline{\mu} = \dfrac{4}{3}\mu \\ \tilde{\rho}_t + (\tilde{\rho}\tilde{v})_r + \dfrac{2}{r}\tilde{\rho}\tilde{v} = 0 \end{cases}
$$

$$
(1.3)'' \qquad \tilde{v}(r,0) = \tilde{v}_0(r), \; \tilde{\rho}(r,0) = \tilde{\rho}_0(r) = \bar{\rho}_0, \; \tilde{v}(0,t) = \tilde{v}(\ell,t) = 0
$$

Here we note that $\tilde{\rho}$ is positive, being expressed as

$$
(1.4) \qquad \tilde{\rho}(r,t) = \overline{\rho}_0\left(\frac{r_0(r,t)}{r}\right)^2 \exp\left[-\int_0^t \tilde{v}_r\left(\overline{r}(r,\tau;t),\tau\right)d\tau\right]
$$

with $\overline{r}(r,\tau;t)$ satisfying the ordinary differential equation

$$
(1.4)' \qquad \frac{d}{d\tau}\overline{r}(r,\tau;t) = \tilde{v}(\overline{r}(r,\tau;t),\tau), \quad \overline{r}(r,t;t) = r
$$

and with $r_0(r,t)$ being defined by

$$
(1.4)'' \qquad r_0(r,t) = \overline{r}(r,0;t)
$$

We note that

(1.5) $$\lim_{r\to 0}\frac{r_0(r,t)}{r}=\frac{\partial}{\partial r}\overline{r}(r,0;t)|_{r=0}=\exp\left[-\int_0^t \tilde{v}_r\,(0,\tau)d\tau\right]>0$$

Hereafter, we shall discuss the problem $(1.2)-(1.2)'$ mainly from a stand point of blowup or non-blowup, while we make in the following sections the Assumption (A) :

Besides $(1.2)'$, v_0 *satisfies* $v_0\in H^{3+\alpha}(I_\ell)$.

2. Fundamental lemmas

We prepare some lemmas in order to discuss our problem.

Lemma 2.1. *Let another condition on* $\tilde{v}_0(r)$ *be added to* $(1.2)'$, *i.e.,*

(2.1) $$\tilde{v}_0(r)\le 0$$

and let for some $T\in(0,\infty)$, $(v,\rho)\in\left(H^{2+\alpha,1+\alpha/2}(O_{\ell,T})\right)^3\times B^{1+\alpha,1+\alpha/2}(O_{\ell,T})$ *satisfy* $(1.2)-(1.2)'$, *it holds that*

(2.2) $$0\ge\tilde{v}(r,t)\ge-|\tilde{v}_0(r)|_{I_\ell}^{(0)}$$

Proof. If $\tilde{v}(r,t)$ takes its positive maximum value at $(r_1,t_1)\in(0,\ell)\times(0,T]$, then it holds that

(2.3) $$0\le(\tilde{\rho}\tilde{v}_t-\overline{\mu}\tilde{v}_{rr})(r_1,t_1)=-\frac{2\overline{\mu}}{r_1^2}\tilde{v}(r_1,t_1)<0$$

which is a contradiction. On the other hand, if $\tilde{v}(r,t)$ takes its negative minimum value at $(r_2,t_2)\in(0,\ell)\times(0,T]$, then it holds that

(2.3)′ $$0\ge(\tilde{\rho}\tilde{v}_t-\overline{\mu}\tilde{v}_{rr})(r_2,t_2)=-\frac{2\overline{\mu}}{r_2^2}\tilde{v}(r_2,t_2)>0$$

which is a contradiction. Thus, by $(1.2)'$, (2.1), (2.3) and $(2.3)'$ we have our assertion. □

From Lemma 2.1 follows that, if blow-up occurs in $(1.2)-(1.2)'$ or $(1.3)'-(1.2)'$, then it does in $(\rho,\nabla v)$ or $(\tilde{\rho},\tilde{v}_r)$. In order to consider this, we introduce $\psi(r,t)$ defined by

(2.4) $$\psi(r,t)=-\frac{\tilde{v}}{r},\quad\psi(0,t)=-\tilde{v}_r(0,t)$$

which, as easily seen, satisfies

(2.5) $$\psi_t(r,t)+\tilde{v}(r,t)\psi_r=\frac{\overline{\mu}}{\tilde{\rho}}\left(\psi_{rr}+4\frac{\psi_r}{r}\right)+\psi^2\ \ (0\le r\le\ell,0\le t\le T)$$

$$(2.5)' \qquad \begin{cases} \psi(r,0) = \psi_0(r) = -\dfrac{\tilde{v}_0(r)}{r} \\ \psi_r(0,t) = \tilde{v}_{rr}(0,t) = \psi(\ell,t) = 0 \\ \psi_0(0) = -\tilde{v}_0'(0) \end{cases}$$

where $(\tilde{\rho}, \tilde{v})$ is the same as in Lemma 2.1.

Lemma 2.2. *Let $\psi_0(r)$ be non-negative and the premise be the same as in Lemma2.1. Then $\psi(r,t)$ is non-negative.*

Proof. If $\psi(r,t)$ takes its negative minimum value at $(r_1, t_1) \in (0,\ell) \times (0,T]$, then

$$(2.6) \qquad 0 \geq \left(\psi_t - \frac{\overline{\mu}}{\tilde{\rho}}\psi_{rr}\right)(r_1,t_1) = \psi(r_1,t_1)^2 > 0$$

which is contradictory. □

Remark 2.1. Assumption (A) guarantees the existence of $\psi_{rr}(0,t)$, which is equal to $\lim_{r\to 0}\frac{\psi_r}{r}$ and $-\tilde{v}_{rrr}(0,t)$.

Lemma 2.3. *Let $\psi(r,t)$ satisfy* $(2.5)-(2.5)'$. *Then $w(r,t)$ defined by $w(r,t) = r^4\psi_r(r,t)$ satisfies the following equation :*

$$(2.7) \qquad w_t(r,t) = \frac{\overline{\mu}}{\tilde{\rho}}w_{rr} + \left\{\left(\frac{\overline{\mu}}{\tilde{\rho}}\right)_r - \frac{4}{r}\frac{\overline{\mu}}{\tilde{\rho}} + r\psi\right\}w_r + (r\psi_r - \psi)w$$

$$(2.7)' \qquad w(r,0) = r^4\psi_0'(r), \quad w(0,t) = 0, \quad w(\ell,t) = \ell^4\psi_r(\ell,t)$$

Proof. Differentiating both sides of (2.5) and suitably substituting $\psi_r = r^{-4}w$ into the resulting equation, we have (2.7). □

Lemma 2.4. *Let $\psi(r,t)$ be as in the lemma above, yet with additional conditions on $\psi_0(r)$*, , i.e.,

$$(2.8) \qquad \psi_0(0) > 0, \quad \psi_0'(r) \leq 0$$

Then we have

$$(2.9) \qquad \psi_r(r,t) \leq 0 \quad (0 \leq r \leq \ell, 0 \leq t \leq T)$$

Proof. We define $W(r,t)$ by $W(r,t) = e^{\lambda t}w(r,t)$, where λ is a constant such that $\lambda > |r\psi_r - \psi|^{(0)}_{O_{\ell,T}}$. Then $W(r,t)$ satisfies

$$(2.10) \qquad W_t(r,t) = \frac{\overline{\mu}}{\tilde{\rho}}W_{rr} + \left\{\left(\frac{\overline{\mu}}{\tilde{\rho}}\right)_r - \frac{4}{r}\frac{\overline{\mu}}{\tilde{\rho}} + r\psi\right\}W_r + \{(r\psi_r - \psi) - \lambda\}w$$

and

$$(2.10)' \qquad W(r,0) = r^4\psi_0'(r), \quad W(0,t) = 0, \quad W(\ell,t) \le 0$$

If $W(r,t)$ takes its positive maximum at $(r_1,t_1) \in (0,\ell) \times (0,T]$, then

$$(2.11) \qquad 0 \le \left(W_t - \frac{\overline{\mu}}{\tilde{\rho}} W_{rr}\right)(r_1,t_1) = \{(r\psi_r - \psi - \lambda)W\}(r_1,t_1) < 0$$

which is a contradiction, and if $W(r,t)$ takes its negative minimum at $(r_2,t_2) \in (0,\ell) \times (0,T]$, then

$$(2.11)' \qquad 0 \ge \left(W_t - \frac{\overline{\mu}}{\tilde{\rho}} W_{rr}\right)(r_2,t_2) = \{(r\psi_r - \psi - \lambda)W\}(r_2,t_2) > 0$$

which is a contradiction. Hence, we have an inequality $w(r,t) = r^4\psi_r(r,t) \le 0$, obtaining (2.9). □

Remark 2.2. We note that there holds a relation

$$(2.12) \qquad 0 \ge w(r,t) = r^4\psi_r(r,t) \ge -|r^4\psi_0'(r)|_{I_\ell}^{(0)} \quad (0 \le r \le \ell)$$

which follows from the behavior of $\psi(r,t)$ in the equality (2.5) near $r = \ell, \ 0 < t \le T$,

$$(2.12)' \qquad 0 = \frac{\overline{\mu}}{\tilde{\rho}}\left(\psi_{rr} + 4\frac{\psi_r}{r}\right)\Big|_{r=\ell}$$

i.e.

$$(2.12)'' \qquad \psi_{rr}|_{r=\ell} = -4\frac{\psi_r}{r}\Big|_{r=\ell} \ge 0$$

The equality $(2.12)''$ guarantees $w(r,t)$ does not take its negative minimum at $r = \ell, \ 0 < t \le T$, finally leading us to (2.12).

Now, let $\tilde{v}, \tilde{\rho}$ and ψ be as in the preceding lemmas. Here, we express $(1.3) - (1.3)'$ by the $\tilde{v}$ - characteristic coordinates $(r_0 = \overline{r}(r,0;t), t_0 = t), \ (0 \le r_0 \le \ell, \ 0 \le t_0 \le T)$ in the following way :

$$\varphi(r_0,t_0) = \frac{r_0}{r(r_0,t_0)}, \quad \hat{\rho}(r_0,t_0) = \tilde{\rho}(r(r_0,t_0),t_0), \quad \hat{v}(r_0,t_0) = \tilde{v}(r(r_0,t_0),t_0)$$

$$(2.13) \qquad \begin{cases} \overline{\rho}_0\varphi^2\hat{v}_{t_0}(r_0,t_0) = \overline{\mu}\left(\dfrac{\hat{v}_{r_0}}{r_{r_0}} + \dfrac{2}{r}\hat{v}\right)_{r_0} \\ \hat{\rho}(r_0,t_0) = \overline{\rho}_0\varphi^2\dfrac{1}{r_{r_0}} \end{cases}$$

$$(2.13)' \qquad \hat{v}(r_0,0) = \tilde{v}_0(r_0), \quad \hat{\rho}(r_0,0) = \overline{\rho}_0, \quad \hat{v}(0,t_0) = \hat{v}(\ell,t_0) = 0$$

In the same way, $(2.5) - (2.5)'$ are expressed as below :

(2.14)
$$\begin{cases} \hat{\psi}(r_0, t_0) = \psi(r(r_0, t_0), t_0) \\ \varphi^2 \hat{\psi}_{t_0} = c_0 \left\{ \left(\dfrac{\hat{\psi}_{r_0}}{r_{r_0}} \right)_{r_0} + 4 \dfrac{\hat{\psi}_{r_0}}{r} \right\} + \varphi^2 \hat{\psi}^2, \quad c_0 = \dfrac{\overline{\mu}}{\bar{\rho}_0} \\ \varphi_{t_0} = \varphi \hat{\psi} \end{cases}$$

(2.14)′
$$\hat{\psi}(r_0, 0) = \psi_0(r_0), \quad \hat{\psi}(0, t_0) = \hat{\psi}(\ell, t_0) = 0, \quad \varphi(r_0, 0) = 1$$

From $(2.14) - (2.14)'$, we have

(2.15)
$$\varphi(r_0, t_0) = \exp\left\{ \int_0^{t_0} \hat{\psi}(r_0, \tau) d\tau \right\}$$

Lemma 2.5. *For* r_{r_0} *in* (2.13), *we have*

(2.16)
$$r_{r_0}(r_0, t_0) = \varphi(0, t_0)^{-3} \varphi(r_0, t_0)^2 \exp\{S(r_0, t_0)\}$$

where

(2.16)′
$$S(r_0, t_0) = \int_0^{r_0} c_0^{-1} \left\{ \varphi^2 \hat{v}(r_0, t_0) - \tilde{v}_0(r_0) - \int_0^{t_0} (\varphi^2)_{t_0} \hat{v}\, dt_0 \right\} dr_0$$

Proof. We note that

(2.17)
$$\overline{\rho}_0 \varphi^2 \hat{v}_{t_0}(r_0, t_0) = \overline{\mu} \left(\frac{\hat{v}_{r_0}}{r_{r_0}} + \frac{2\hat{v}}{r} \right)_{r_0} = \overline{\mu} \{ \log(r_{r_0} r^2) \}_{r_0 t_0}$$

By integration in t_0, we have

(2.17)′
$$\begin{aligned} \int_0^{t_0} c_0^{-1} \varphi^2 \hat{v}_{t_0} dt_0 &= c_0^{-1} \left\{ \varphi^2 \hat{v}(r_0, t_0) - \tilde{v}_0(r_0) - \int_0^{t_0} (\varphi^2)_{t_0} \hat{v}\, dt_0 \right\} \\ &= \{ \log(r_{r_0} r^2) \}_{r_0} \Big|_{t_0=0}^{t_0} = \{ \log(r_{r_0} r^2) \}_{r_0} - \{ \log(r_0^2) \}_{r_0} \end{aligned}$$

Nextly, by integration in r_0 over $[\varepsilon, r_0]$, it holds that

(2.17)″
$$\begin{aligned} &\int_\varepsilon^{r_0} c_0^{-1} \left\{ \varphi^2 \hat{v}(r_0, t_0) - \tilde{v}_0(r_0) - \int_0^{t_0} (\varphi^2)_{t_0} \hat{v}\, dt_0 \right\} dr_0 \\ &= \int_\varepsilon^{r_0} \{ \log(r_{r_0} r^2) - \log(r_0^2) \}_{r_0} dr_0 = \log(r_{r_0} r^2 r_0^{-2}) |_{r_0=\varepsilon}^{r_0} \\ &= \log(r_{r_0} \varphi^{-2}) - \log(r_{r_0} \varphi^{-2}) |_{r_0=\varepsilon} \end{aligned}$$

By $\varepsilon \to 0$, we obtain

(2.17)‴
$$S(r_0, t_0) = \log(r_{r_0} \varphi^{-2}) - \log(\varphi(0, t_0)^{-3})$$

from which comes (2.16). □

Lemma 2.6. *The estimate*

$$0 \le -\int_0^{t_0} \hat{\psi}_{r_0}(r_0,\tau)d\tau \le a\ell e^{a\ell^2} \tag{2.18}$$

holds, where

$$a = \frac{c_0^{-1}}{2}\left(\psi_0(0) + \varphi(0,t_0)^2|\hat{\psi}(0,t_0)| + 2\int_0^{t_0} \varphi(0,\tau)^2\hat{\psi}(0,\tau)^2 d\tau\right) \tag{2.18$'$}$$

Proof. From $\varphi(r_0,t_0) = r_0/r(r_0,t_0)$, $(2.14)_3$

$$r_{r_0} = \left(\frac{r_0}{\varphi}\right)_{r_0} = \frac{1}{\varphi}\left(1 - r_0\int_0^{t_0} \hat{\psi}_{r_0}(r_0,\tau)d\tau\right) \tag{2.19}$$

is obtained. Hence, from Lemma 2.5 and (2.19), we have the equality

$$\begin{aligned} -\int_0^{t_0} \hat{\psi}_{r_0}(r_0,\tau)d\tau &= r_0^{-1}(\varphi r_{r_0} - 1) \\ &= r_0^{-1}(\varphi(0,t_0)^{-3}\varphi^3(r_0,t_0)\exp\{S(r_0,t_0)\} - 1) \end{aligned} \tag{2.20}$$

Because of Lemma 2.4, $(2.14)_3$ and (2.15) , we find that $S(r_0,t_0) \le \int_0^{r_0} 2ar_0 dr_0 = ar_0^2$. Consequently, we obtain

$$\begin{aligned} 0 \le -\int_0^{t_0} \hat{\psi}_{r_0}(r_0,\tau)d\tau &\le r_0^{-1}\left(e^{ar_0^2} - 1\right) \\ &= ar_0\left(1 + \sum_{n=2}^{\infty}\frac{a^n r_0^{2(n-1)}}{n!}\right) \le ar_0 e^{ar_0^2} \end{aligned} \tag{2.21}$$

□

Lemma 2.7. $r_{r_0 r_0}$ *is expressed as below* :

$$r_{r_0 r_0} = \varphi(0,t_0)^{-3}\varphi(r_0,t_0)^3\exp\{S(r_0,t_0)\}\left\{\int_0^{t_0} \hat{\psi}_{r_0}(r_0,\tau)d\tau + S_{r_0}(r_0,t_0)\right\} \tag{2.22}$$

where S_{r_0} *is such that*

$$S_{r_0}(r_0,t_0) = c_0^{-1}\left\{\varphi^2\hat{v}(r_0,t_0) - \tilde{v}_0(r_0) + \int_0^{t_0}(\varphi^2)_{t_0}\hat{v}\,dt_0\right\} \tag{2.22$'$}$$

from which follows an easy estimation of $|S_{r_0}|$.

Proof. The assertion of the lemma comes directly from Lemma 2.5 and Lemma 2.6. □

Lemma 2.8. *Let $\tilde{v}$ be as in Theorem 1.1. Then, it holds that*

$$\tilde{v}_r(\ell, t) \leq \sup_{0\leq r\leq \ell} \tilde{v}_0'(r) \equiv k_0 \tag{2.23}$$

Proof. We put $V(r.t)$

$$V(r,t) = \tilde{v}(r,t) - k_0 r \ (\leq 0) \tag{2.24}$$

Taking account of (1.3) − (1.3)', we have a relation

$$\begin{aligned} \tilde{\rho}(r,t)\{V_t + \tilde{v}(V_r + k_0)\} &= \overline{\mu}\left\{V_{rr} + \frac{2}{r}(V_r + k_0) - \frac{2}{r^2}(V + k_0 r)\right\} \\ &= \overline{\mu}\left(V_{rr} + \frac{2}{r}V_r - \frac{2}{r^2}V\right) \end{aligned} \tag{2.25}$$

from which comes an equality

$$\tilde{\rho}(r,t)V_t = \overline{\mu}\left(V_{rr} + \frac{2}{r}V_r - \frac{2}{r^2}V\right) - \tilde{\rho}\tilde{v}(V_r + k_0) \tag{2.25$'$}$$

where we remark that

$$\begin{cases} V(\ell,t) = -k_0\ell, \quad V(0,t) = 0 \\ 0 \geq V(r,0) = \tilde{v}(r,0) - k_0 r \geq -k_0\ell \end{cases} \tag{2.25$''$}$$

By the maximum value principle, $V(r.t)$ takes its minimum value at $r = \ell \ (t > 0)$. Therefore, on the basis of its non-positivity, it holds that

$$V_r(\ell,t) = \tilde{v}_r(\ell,t) - k_0 \leq 0 \tag{2.26}$$

which is equivalent to (2.23). □

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