Abstract

Gauss’ and Euler’s type infinite product representations for Vignéras multiple gamma function are presented. As an application of the representations, a multiplication formula for the function is derived.

Key words: multiple gamma function, Gauss product representation, Euler product representation, multiplication formula

1. Introduction

In a series of papers [2, 3, 4, 5], Barnes introduced multiple gamma functions associated with a certain generalization of the Hurwitz zeta function. In relevant with a special case of Barnes’ function, Vignéras [15] introduced her multiple gamma functions $G_r(z)$ ($r \in \mathbb{Z}_{\geq 0}$) as a sequence of meromorphic functions uniquely determined by the following relations:

\begin{align}
(i) & \quad G_0(z) = z, \\
(ii) & \quad G_r(1) = 1, \\
(iii) & \quad G_r(z + 1) = G_{r-1}(z)G_r(z)
\end{align}

\begin{align}
(iv) & \quad \frac{d^{r+1}}{dz^{r+1}} \log G_r(z + 1) \geq 0 \quad \text{for } z \geq 0.
\end{align}

This formulation can be considered as a generalization of the Bohr-Morellup theorem. For example, $G_1(z)$ is the celebrated Euler gamma function $\Gamma(z)$ (cf. Artin [1], Whittaker-Watson [16]). $G_2(z)$ is $G$-function introduced in Barnes [2].

In this paper, we present two types of infinite product representations of Vignéras’ multiple gamma function, which can be considered as a generalization of the Gauss and the Euler product formula of Euler’s gamma function

\begin{align}
\Gamma(z + 1) &= \lim_{N \to \infty} \frac{N!}{(z+1)(z+2)\cdots(z+N)}(N+1)^z \\
&= \prod_{n=1}^{\infty} \left[ \left(1 + \frac{z}{n} \right)^{-1} \left(1 + \frac{1}{n} \right)^z \right]
\end{align}

(cf. Artin [1], Whittaker-Watson [16]). Our main theorem is stated as follows: If $z$ is not negative
integer, the multiple gamma function $G_r(z)$ is represented as

$$G_r(z+1) = \lim_{N \to \infty} \left[ \prod_{n=1}^{N} \frac{G_{r-1}(n)}{G_{r-1}(z+n)} \prod_{k=0}^{r-1} G_k(N+1) \left( r-\frac{z}{r-k} \right) \right]$$ (4)

$$= \prod_{n=1}^{\infty} \left[ \frac{G_{r-1}(n)}{G_{r-1}(z+n)} \prod_{k=0}^{r-1} \left( \frac{G_k(n+1)}{G_k(n)} \right) \left( r-\frac{z}{r-k} \right) \right].$$ (5)

In the case when $r = 1$, these formulas coincide with (2) and (3). We can find the representation for $G_2(z)$ in Jackson [6]. It should be noted that infinite product formulas of these types for a q-analogue of the multiple gamma function were already obtained in [12]. However, in contrast to simplicity in q-case, some delicate techniques are necessary to deal with infinite products of Vignéras’ function. We verify (4) and (5) in section 1. The point is to apply an asymptotic expansion in [13] to estimations for products of Vignéras’ functions.

In section 2, as an application of infinite product representations, we derive a multiplication formula for Vignéras’ multiple gamma function, which can be regarded as a generalization of the well known formula

$$\prod_{m=0}^{p} \Gamma \left( \frac{z + m}{p} \right) = \frac{(2\pi)^{\frac{p-1}{2}}}{p^{\frac{p}{2}}} \Gamma(z)$$ (6)

for Euler’s gamma function (cf. Artin [1], Whittaker-Watson [16]). It is described as follows:

$$\prod_{q_1, q_2, \cdots, q_r = 0}^{p-1} G_r \left( \frac{z + q_1 + \cdots + q_r}{p} \right) = \frac{e^{\phi_r(z)}}{p^{\psi_r(z)}} G(z)$$

It might be seem that formula of this type can be guessed easily from (1). However, it is not easy to determine explicit forms of $\phi_r(z)$ and $\psi_r(z)$. The reason why we can do it is usefulness of our representations (4).

For simplicity, we call Vignéras multiple gamma function only “multiple gamma function” in the following sections.

Notations: In this paper, we use notation $B_r(z)$ for the Bernoulli polynomial defined by the generating function

$$\sum_{r=0}^{\infty} B_r(z) t^r = \frac{te^t}{1-e^t},$$

and $B_r$ for the Bernoulli number defined as $B_r := B_r(0)$. We introduce the Stirling number $rS_j$ of the 1 st kind by

$$t(t-1) \cdots (t-r+1) = \sum_{j=0}^{r} rS_j t^j.$$

The notation $\zeta(s)$ is used to refer to the Riemann zeta function defined as the series $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$ and its analytical continuation. $\zeta'(s)$ is the first derivative of $\zeta(s)$ defined by $\zeta'(s) := \frac{d}{ds} \zeta(s)$.
2 Infinite product representations

As mentioned in introduction, our main theorem is described as follows:

**Theorem 2.1** If $z$ is not negative integer and is included in any finite region of complex plane, the multiple gamma function $G_r(z)$ is represented as

$$G_r(z + 1) = \lim_{N \to \infty} \left[ \prod_{n=1}^{N} \frac{G_{r-1}(n)}{G_{r-1}(z + n)} \prod_{k=0}^{r-1} G_k(N + 1)^{\left(\frac{z}{r - k}\right)} \right]$$

(7)

$$= \prod_{n=1}^{\infty} \left[ \frac{G_{r-1}(n)}{G_{r-1}(z + n)} \prod_{k=0}^{r-1} \left( \frac{G_k(n)}{G_k(n)} \right)^{\left(\frac{z}{r - k}\right)} \right].$$

(8)

**Proof.** From the Gauss product representation (7), the Euler product representation (8) follows immediately. So, we give a proof of (7) in this section. We apply an asymptotic expansion for $G_r(z)$, which was firstly appeared in [13].

**Theorem 2.2** (Ueno-Nishizawa) Let us put $0 < \delta < \pi$, then, as $|z| \to \infty$ in the sector $\{ z \in \mathbb{C} \mid \arg z < \pi - \delta \}$,

$$\log G_r(z + 1) = \left\{ \begin{array}{l}
\left( \frac{z + 1}{r} \right) + \sum_{j=0}^{r-1} \frac{B_{j+1}}{j+1} G_{r,j}(z) \end{array} \right\} \log(z + 1) -
- \sum_{j=0}^{r-1} G_{r,j}(z) \frac{(z + 1)^{j+1}}{(j + 1)^2} - \sum_{j=0}^{r-1} G_{r,j}(z) \zeta'(-j) + O(z^{-1}).$$

(9)

where a polynomial $G_r,j(z)$ is dened by the generating function

$$\left( \frac{z - u}{r - 1} \right) =: \sum_{j=0}^{r-1} G_{r,j}(z) u^j \quad (r = 0 \cdots r - 1), \quad G_{r,j}(z) = 0, \quad (j \geq r).$$

In our proof, the following lemma is useful:

**Lemma 2.3** For arbitrary $x, y \in \mathbb{C}$,

(i) $\sum_{k=0}^{r} \left( \begin{array}{c}
x \\
r - k \end{array} \right) \left( \begin{array}{c}
y \\
k \end{array} \right) = \left( \begin{array}{c}
x + y \\
r \end{array} \right)$,  
 (ii) $\sum_{k=0}^{r} \left( \begin{array}{c}
x \\
r - k \end{array} \right) G_{k,j}(y) = G_{r,j}(x + y)$.

Noting this lemma and that

$$\sum_{j=0}^{r-1} G_{r,j}(z + N - 1) \left\{ \frac{(z + N)^{j+1}}{j + 1} - \frac{N^{j+1}}{(j + 1)^2} \right\} = \int_{N}^{z + N} \frac{du}{v} \int_{0}^{v} \left( \begin{array}{c}
z + N - 1 - u \\
r - 1 \end{array} \right) du,$$

we rewrite the logarithms of terms in brackets of (7) and have the following asymptotic behavior as $N \to \infty$:

$$\log \left[ \prod_{n=1}^{N} \frac{G_{r-1}(n)}{G_{r-1}(z + n)} \prod_{k=0}^{r-1} G_k(N + 1)^{\left(\frac{z}{r - k}\right)} \right] =$$
As $N \to \infty$, this integral vanishes because of the following lemma, which was already shown in [13]:

**Lemma 2.4 (Ueno-Nishizawa)** For arbitrary $z \in \mathbb{C}$, we have

\[
\binom{z}{r} + \sum_{j=0}^{r-1} \frac{B_{j+1}}{j+1} G_{r,j}(z-1) = \int_{-1}^{z} \binom{u}{r-1} du + \int_{u}^{z} \binom{z-1-u}{r-1} du + O(N^{-1}).
\]

Therefore, we have proved theorem 2.1.

### 3 Multiplication formula

As an application of Gauss’ product representation, we demonstrate the multiplication formula of the multiple gamma function.

**Theorem 3.1**

\[
\prod_{q_{1}, q_{2}, \cdots, q_{r}=0}^{p-1} G_{r} \left( \frac{z + q_{1} + \cdots + q_{r}}{p} \right) = e^{\phi_{r}(z)} G(z) \tag{10}
\]

where

\[
\phi_{r}(z) = \sum_{j=0}^{r-1} \left[ \sum_{q_{1}, \cdots, q_{r}=0}^{p-1} G_{r,j} \left( \frac{z + q_{1} + \cdots + q_{r}}{p} - 2 \right) - G_{r,j}(z-1) \right] \zeta'(-j)
\]

\[
\psi_{r}(z) = \binom{z}{r} + \sum_{j=0}^{r-1} \frac{B_{j+1}}{j+1} G_{r,j}(z-1).
\]

**Proof.** From the infinite product representation (7), it follows that
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We substitute the asymptotic expansion (9) to the logarithm of terms in the second brackets.

\[
\log \left[ \prod_{k=0}^{r} \frac{G_r(N) \sum_{q_1, \ldots, q_r} \left( \frac{(z + q_1 + \cdots + q_r)}{r} \right)}{G_r(p(N - 1))^{\left(\frac{z}{r-1}\right)}} \times \frac{\sum_{m=0}^{p(N-1)-1} \psi_{r-1}(z + m)}{e^{\sum_{m=0}^{p(N-1)-1} \phi_{r-1}(z + m)}} \right] =
\]

\[
= \left\{ \sum_{q_1, \ldots, q_r} \left( \frac{z + q_1 + \cdots + q_r}{r} \right) + \frac{\sum_{m=0}^{p(N-1)-1} \psi_{r-1}(z + m)}{e^{\sum_{m=0}^{p(N-1)-1} \phi_{r-1}(z + m)}} \right\} \log N -
\]

\[
- \left\{ \sum_{j=0}^{r} \left( \frac{z + p(N - 1)}{r} \right) + \frac{\sum_{m=0}^{p(N-1)-1} \psi_{r-1}(z + m)}{e^{\sum_{m=0}^{p(N-1)-1} \phi_{r-1}(z + m)}} \right\} \log (N - 1) -
\]

\[
- \sum_{j=0}^{r} \left\{ \sum_{q_1, \ldots, q_r} \frac{G_r(z + q_1 + \cdots + q_r)}{r} + \frac{\sum_{m=0}^{p(N-1)-1} \psi_{r-1}(z + m)}{e^{\sum_{m=0}^{p(N-1)-1} \phi_{r-1}(z + m)}} \right\} \log p -
\]

\[
- \sum_{j=0}^{r} \left\{ \sum_{q_1, \ldots, q_r} \frac{G_r(z + q_1 + \cdots + q_r)}{r} + \frac{\sum_{m=0}^{p(N-1)-1} \psi_{r-1}(z + m)}{e^{\sum_{m=0}^{p(N-1)-1} \phi_{r-1}(z + m)}} \right\} \zeta(-j) -
\]

\[
- \sum_{m=0}^{p(N-1)-1} \phi_{r-1}(z + m) + o(1).
\]

We show that its divergent terms vanish. First, we compute terms including \( \log p \).

**Proposition 3.2** If we define \( \psi_0(z) = 0 \) and

\[
\psi_r(z) := \left( \frac{z}{r} \right) + \frac{\sum_{j=0}^{r-1} B_{j+1} G_{r,j}(z - 1)}{j + 1},
\]

then \( \psi_r(z) \) satisfies \( \psi_0(z) = z \) and

\[
\left( \frac{p(N - 1) - 1}{r} \right) + \frac{\sum_{j=0}^{r-1} B_{j+1} G_{r,j}(z + p(N - 1) - 2)}{j + 1} - \sum_{m=0}^{p(N-1)-1} \psi_{r-1}(z + m) = \psi_r(z).
\]

\( \psi_r(z) \) does not depend on \( N \) and is uniquely determined as the polynomial satisfying the above recurrence relation.

**Proof.** This proposition immediately follows from the relation
for $L \in \mathbb{Z}_{\geq 0}$.

Next, we simplify terms including $\zeta'(-j)$ and give a explicit form of $\phi_r(z)$.

**Proposition 3.3** If we define
\[
\phi_{r,j}(z) := \sum_{q_1,\ldots,q_r=0}^{p-1} G_{r,j} \left( \frac{z + q_1 + \cdots + q_r}{p} - 2 \right) - G_{r,j}(z - 1),
\]
then $\phi_r(z) = \sum_{j=0}^{r-1} \phi_{r,j}(z) \zeta'(-j)$ is uniquely determined as a polynomial satisfying the recurrence relation $\phi_0(z) = 0$ and
\[
\sum_{j=0}^{r-1} \left[ \sum_{q_1,\ldots,q_r=0}^{p-1} G_{r,j} \left( \frac{z + q_1 + \cdots + q_r}{p} + N - 2 \right) - G_{r,j}(z + p(N-1) - 1) \right] \zeta'(-j) - \sum_{m=0}^{p(N-1)-1} \phi_{r-1}(z + m) = \phi_r(z).
\]

**Proof.** It is sufficient to prove
\[
\phi_{r,j}(z) = \sum_{q_1,\ldots,q_r} G_{r,j} \left( \frac{z + q_1 + \cdots + q_r}{p} + N - 2 \right) - G_{r,j}(z + p(N-1) - 1) - \sum_{m=0}^{p(N-1)-1} \left[ \sum_{q_1,\ldots,q_r} G_{r-1,j} \left( \frac{z + m + q_1 + \cdots + q_r}{p} + N - 2 \right) - G_{r-1,j}(z + m + p(N-1) - 1) \right].
\]

We can see from the identity
\[
\sum_{m=0}^{L} G_{r,j}(z + m) = G_{r+1,j}(z + L) - G_{r+1,j}(z), \quad (L \in \mathbb{Z}_{\geq 0}),
\]
and
\[
\sum_{m=0}^{p(N-1)-1} \sum_{q_1,\ldots,q_{r-1}=0}^{p-1} G_{r,j} \left( \frac{z + m + q_1 + \cdots + q_r}{p} - 2 \right) = \sum_{q_1,\ldots,q_{r-1},q_r=0}^{p-1} G_r \left( \frac{z + q_1 + \cdots + q_r}{p} + N - 2 \right) - G_r \left( \frac{z + q_1 + \cdots + q_r}{p} - 2 \right),
\]
The uniqueness of $\phi_r(z)$ follows from its polynomiality.

In order to finish our proof, we verify that the rest of terms vanish as $N \to \infty$. By lemma 2.3, we can see that
\[
\sum_{k=0}^{r} \left\{ \sum_{q_1,\ldots,q_r=0}^{p-1} \left( \frac{z + q_1 + \cdots + q_r}{r-k} \right) / p - 1 \right\} \times
\]

\[
\times \left[ \binom{N}{k} + \sum_{j=0}^{k-1} \frac{B_{j+1}}{j+1} G_{k,j} (N-1) - \sum_{j=0}^{r} G_{k,j} (N-1) \frac{N^2}{(j+1)^2} \right] - \\
- \sum_{k=0}^{r} \binom{z-1}{r-k} \left( \left( \binom{N}{k} + \sum_{j=0}^{r-1} \frac{B_{j+1}}{j+1} G_{k,j} (N-1) - \sum_{j=0}^{k-1} G_{k,j} (N-1) \frac{N^j+1}{(j+1)^2} \right) \right) \\
= \sum_{q_1, \ldots, q_r=0}^{p-1} \left[ \frac{z + q_1 + \cdots + q_r}{p} - 1 \right] + \sum_{j=0}^{r-1} \frac{B_{j+1}}{j+1} G_{r,j} \left( \frac{z + q_1 + \cdots + q_r}{p} + N - 2 \right) - \\
- \sum_{j=0}^{r-1} G_{r,j} \left( \frac{z + q_1 + \cdots + q_r}{p} + N - 2 \right) \frac{N^2}{(j+1)^2} - \\
- \left\{ \binom{z+N-1}{r} + \sum_{j=0}^{r-1} \frac{B_{j+1}}{j+1} G_{r,j} (z + N - 2) - \sum_{j=0}^{r-1} G_{r,j} (z + N - 2) \frac{N^j+1}{(j+1)^2} \right\}. 
\]

From the same argument as proof of theorem 2.1, it follows that the above terms tend to zero as \( N \to \infty \). Therefore, we have proved theorem 3.1.

Our result is closely related with Kuribayashi [7]. In order to explain his result, we introduce some functions. \( \zeta_r(s, z) \) is defined as a special case of Barnes’ zeta function [5, 14], which is introduced as the series

\[
\zeta_r(s, z) := \sum_{n_1, \ldots, n_r=0}^{\infty} (z + n_1 + \cdots + n_r)^{-s}
\]

for \( \Re s > r \). This function can be continued analytically to a meromorphic function whose poles are placed at \( s = 1, \cdots, r \). We call the analytic continuation also \( \zeta_r(s, z) \). The gamma function \( \Gamma_r(z) \) associated with \( \zeta_r(s, z) \) is introduced as

\[
\Gamma_r(z) := \exp \left[ \frac{\partial}{\partial s} \zeta_r(s, z) \right]_{s=0}.
\]

Kuribayashi exhibit the following multiplication formula:

**Theorem 3.4 (Kuribayashi)** \( \Gamma_r(z) \) satisfies the following multiplication formula:

\[
\prod_{q_1, \ldots, q_r=0}^{p-1} \Gamma_r \left( \frac{z + q_1 + \cdots + q_r}{p} \right) = p^{Q_r(z)} \Gamma_r(z),
\]

where

\[
Q_r(z) = \frac{(-1)^r}{(r-1)!} \sum_{r=1}^{p} \frac{r!}{l} \left\{ z^l - (-1)^l B_l \right\}.
\]

As a consequence of facts in Vardi [14], a relation between \( G_r(z) \) and \( \Gamma_r(z) \) is expressed as follows:
Thus, we have
\[ Q_r(z) = (-1)^r \psi_r(z) = (-1)^r \left( \frac{z}{r} + \sum_{j=0}^{r-1} \frac{B_{j+1}}{j+1} G_{r,j}(z - 1) \right). \] (11)
Our expression is useful in some cases of studies on related functions. For example, noting that \( G_{r,0}(z) = (z,1) \), we can check that the relation follows
\[ (-1)^r Q_r(r-z) = Q_r(z) \] from the definition of \( \psi_r(z) \) and (11). It plays an important role in the multiplication formula
\[ \prod_{q_1, \ldots, q_r = 0}^{r-1} S_r \left( \frac{z + q_1 + \cdots + q_r}{p} \right) = S_r(z). \]
for Kurokawa’s multiple sine function \([8, 9, 10, 11]\) introduced as
\[ S_r(z) := \Gamma_r(r - z) \Gamma_r(z)(-1)^{r+1}. \]
In Kuribayashi’s original proof, (12) is verified through a rather complicated argument, He applied a relation between \( \zeta_r(-m, z) \) \((m \in \mathbb{Z}_0^\geq)\) and the Bernoulli polynomials \( B_l(z) \). However, once (11) is obtained, we can check (12) immediately.

4 Appendix : an elementary proof for (11)
Without facts of zeta functions, we can prove (11) directly as follows: First, we rewrite Kuribayashi’s \( Q_r(z) \) as
\[ (-1)^r Q_r(z) = \frac{1}{(r-1)!} \sum_{l=0}^{r-1} r_{r-1} S_l \left\{ \frac{(-1)^{l+1} B_{l+1}}{l+1} - \frac{(z-1)^{l+1}}{l+1} \right\}. \] (13)
The second term can be written as follows:
\[ \frac{1}{(r-1)!} \sum_{l=0}^{r-1} r_{r-1} S_l \frac{(z-1)^{l+1}}{l+1} = \int_0^z \left( \frac{t-1}{r-1} \right) dt - \int_0^1 \left( \frac{t-1}{r-1} \right) dt. \]
From Lemma 2.4 and
\[ G_{r,j}(0) = \frac{(-1)^j}{(r-1)!} r_{r-1} S_j, \]
it follows that
\[ \int_0^z \left( \frac{t-1}{r-1} \right) dt - \int_0^1 \left( \frac{t-1}{r-1} \right) dt = \frac{z}{r} + \sum_{j=0}^{r-1} \frac{B_{j+1}}{j+1} G_{r,j}(z - 1) - \frac{1}{(r-1)!} \sum_{j=0}^{r-1} \frac{B_{j+1}}{j+1} (-1)^j r_{r-1} S_j. \]
Therefore, we obtain (11) by substituting this to (13).
References