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Initial-Boundary Value Problem for Some Viscous Incompressible Flow

ある非圧縮粘性流に対する初期値境界値問題

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Abstract. We prove the local existence of weak solutions to the initial-boundary value problem with a nonnegative initial density for a nonhomogeneous viscous incompressible fluid.

§1. Introduction. The motion of a nonhomogeneous viscous incompressible fluid is described by the system of equations for the density $\rho(t, x)$, the velocity $v(t, x) = (v^1(t, x), v^2(t, x), v^3(t, x))$, the pressure $p(t, x)$ and the absolute temperature $\theta(t, x)$ (cf. [1]):

$$(1.1)\qquad \begin{cases} \rho_t + (v\cdot\nabla)\rho = 0, \\ \rho[v_t + (v\cdot\nabla)v] + \nabla p = \mu\Delta v + \rho f, \\ \nabla\cdot v = 0, \\ c_V\rho[\theta_t + (v\cdot\nabla)\theta] = \varkappa\Delta\theta + 2\mu D{:}D, \end{cases}$$

where $f(t, x) = (f^1(t, x), f^2(t, x), f^3(t, x))$ is the outer force and D is the velocity deformation tensor with the elements $D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$, $i, j = 1, 2, 3$ and $D{:}D = \sum_{i, j=1}^{3} D_{ij}D_{ij}$. We assume that μ (the coefficient of viscosity), c_V (the specific heat at constant volume) and $\varkappa$ (the coefficient of heat conduction) are positive constants.

Let Ω be a bounded open set in R^3 with a smooth boundary $\partial\Omega$. We consider the above system under the following initial-boundary condition:

$$(1.2)\qquad \begin{cases} (\rho(0, x), v(0, x), \theta(0, x)) = (\rho_0(x), v_0(x), \theta_0(x)), x \in \Omega, \\ v|_{\partial\Omega} = 0, \ \theta|_{\partial\Omega} = \bar{\theta}, \ t > 0, \end{cases}$$

where $\bar{\theta}$ is a fixed positive constant and we always assume that the compatibility conditions are satisfied. We note that the problem (1.1), (1.2) is rewritten to the problem (1.1),

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$$(1.3)\qquad \begin{cases} (\rho(0,x),\ v(0,x),\ \theta(0,x)) = (\rho_0(x),\ v_0(x),\ \theta_0(x)-\bar{\theta}), \\ (v,\ \theta)|_{\partial\Omega}=0,\ t>0, \end{cases}$$

by the change of variables $(\rho, v, \theta)\to(\rho, v, \theta+\bar{\theta})$. Let us term the problem (1.1), (1.3) IBP.

We are much interested in the case of $\rho_0\geqq 0$, because in the usual result, it seems that the assumption that $\rho_0(x)$ is strictly positive is essential.

In this paper we show the existence of a weak solution to IBP, in the sense of Definition in §2, under suitable assumptions.

§2. Statement of Result. For $s\in R$, $H^S(\Omega)$ denotes the usual Sobolev space. Let us introduce $C_\sigma^\infty(\Omega)=\{u\in C_0^\infty(\Omega)^3;\ \nabla\cdot u=0\}$ and $H_\sigma^S(\Omega)=$closure of $C_\sigma^\infty(\Omega)$ in $H^S(\Omega)^3$.

Next we give the definition of a weak solution to IBP.

Definition. The functions $\rho(t,x)$, $v(t,x)=(v^1(t,x),\ v^2(t,x),\ v^3(t,x))$ and $\theta(t,x)$ called a weak solution to IBP, if $\rho(t,x)\in L^\infty([0,T]\times\Omega)$, $v(t,x)\in L^2(0,T;\ H_\sigma^1(\Omega))$ and $\theta(t,x)\in L^2(0,T;\ H_0^1(\Omega))$ and if the integral identities

$$(2.1)\qquad \int_0^T\int_\Omega(\rho\varphi_t+\sum_{j=1}^3\rho v^j\varphi_{x_j})\,dxdt+\int_\Omega\rho_0(x)\varphi(0,x)\,dx=0,$$

$$(2.2)\qquad \int_0^T\int_\Omega(\rho v\cdot\Phi_t+\sum_{j=1}^3\rho v^j v\cdot\Phi_{x_j}-\mu\sum_{j=1}^3 v_{x_j}\cdot\Phi_{x_j}+\rho f\cdot\Phi)\,dxdt + \int_\Omega\rho_0(x)v_0(x)\cdot\Phi(0,x)\,dx=0$$

and

$$(2.3)\qquad \int_0^T\int_\Omega(c_V\rho\theta\varphi_t+c_V\sum_{j=1}^3\rho v^j\theta\varphi_{x_j}-\kappa\sum_{j=1}^3\theta_{x_j}\varphi_{x_j}-2\mu D{:}D\varphi)\,dxdt + \int_\Omega c_V\rho_0(x)(\theta_0(x)-\bar{\theta})\varphi(0,x)\,dx=0$$

hold for any $\varphi\in C^1(0,T;\ H^1(\Omega))$ and $\Phi\in C^1(0,T;\ H_\sigma^1(\Omega))$ such that $\varphi(T,x)=0$ and $\Phi(T,x)=0$.

Now we can state our result.

Theorem. *Suppose that* $0\leqq\rho_0(x)\leqq M$, $v_0(x)\in H_\sigma^1(\Omega)$, $\theta_0(x)-\bar{\theta}\in L^2(\Omega)$ *and* $f(t,x)\in L^2(0,T;\ L^2(\Omega)^3)$. *Then there exists a weak solution for some* $T'\in(0,T]$ *such that* $\rho(t,x)$

$\in L^{\infty}([0, T'] \times \Omega)$, $v(t, x) \in L^{\infty}(0, T'; H^1_{\sigma}(\Omega)) \cap L^2(0, T'; H^2(\Omega))$ *and* $\theta(t, x) \in L^2(0, T'; H^1_0(\Omega))$.

The proof is given in §§3-7.

§3. Proof of Theorem (first step). Let P be the projection of $L^2(\Omega)^3$ onto $H^0_{\sigma}(\Omega)$ and let us consider the eigenvalue problem:

$$
(3.1) \qquad \begin{cases} \mu P\Delta\psi_k + \lambda_k \psi_k = 0 \ \text{in } \Omega, \\ \psi_k|_{\partial\Omega} = 0. \end{cases}
$$

With respect to the properties of the operator $\mu P\Delta$, we refer to [2]. For example,

Lemma 3.1. *$\mu P\Delta$ is a self-adjoint operator in $H^0_{\sigma}(\Omega)$ and its inverse is compact.*

Therefore we find that

$$
(3.2) \qquad 0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_k \to \infty \ \text{as } k \to \infty
$$

and

(3.3) $\{\psi_k\}_{k=1}^{\infty}$ is a orthonormal system in $H^0_{\sigma}(\Omega)$.

Let $\{\rho_{0m}(x)\}_{m=1}^{\infty}$ be a sequence of functions such that $\rho_{0m}(x) \in C^1(\bar{\Omega})$, $\frac{1}{m} \leqq \rho_{0m}(x) \leqq M + \frac{1}{m}$ and $\rho_{0m}(x) \to \rho_0(x)$ in $L^2(\Omega)$ and let $\tilde{v}_m(t, x) = \sum_{k=1}^{m} \tilde{a}_{mk}(t)\psi_k(x)$, where $\tilde{a}_{mk}(t) \in C^0([0, T])$, $k = 1, \cdots, m$.

Now we consider the following Cauchy problem:

$$
(3.4) \qquad \begin{cases} \rho_{mt} + \sum_{j=1}^{3} \tilde{v}^j_m \rho_{mx_j} = 0 \\ \rho_m(0, x) = \rho_{0m}(x). \end{cases}
$$

Then we have

Lemma 3.2. *There exists a number* $T_1 \in (0, T]$ *such that* (3.4) *has a unique solution*

$\rho_m(t, x) \in C^1([0, T_1] \times \bar{\Omega})$ *satisfying* $\frac{1}{m} \leqq \rho_m(t, x) \leqq M + \frac{1}{m}$.

Proof. For any $(t, x) \in [0, T] \times \bar{\Omega}$, we consider a system of ordinary differential equations:

$$\left\{ \begin{array}{l} \frac{d}{d\tau}\bar{x}(\tau; t, x) = \tilde{v}_m(\tau, \bar{x}(\tau; t, x)), \\ \bar{x}(t; t, x) = x, \ 0 \leqq \tau \leqq t. \end{array} \right. \tag{3.5}$$

Then there exists a number $T_1 \in (0, T]$ such that (3.5) has a unique solution curve $\bar{x}(\tau; t, x) \in C^1([0, T_1]); C^1(\bar{\Omega})^3)$ passing (t, x). If we put $x_0(t, x) = \bar{x}(0; t, x)$, then

$$\rho_m(t, x) = \rho_{0m}(x_0(t, x)) \tag{3.6}$$

is a desired solution. Q.E.D.

§4. Proof of Theorem (second step). Let $\{f_m(t, x)\}_{m=1}^{\infty}$ be a sequence of functions such that $f_m(t, x) \in C^1([0, T_1]; L^2(\Omega)^3)$ and $f_m(t, x) \to f(t, x)$ in $L^2(0, T_1; L^2(\Omega)^3)$ and let us consider the Cauchy problem for a system of ordinary differential equations:

$$\left\{ \begin{array}{l} \sum_{k=1}^{m} \alpha_{jk}^m(t) \frac{d}{dt} a_{mk}(t) + \sum_{k,\ l=1}^{m} \beta_{jkl}^m(t) a_{mk}(t) a_{ml}(t) + \lambda_j a_{mj}(t) \\ \qquad\qquad = \gamma_j^m(t), \ j = 1, \cdots, m, \\ a_{mk}(0) = \int_{\Omega} v_0(x) \cdot \psi_k(x) dx, \ k = 1, \cdots, m, \end{array} \right. \tag{4.1}$$

where

$$\alpha_{jk}^m(t) = \int_{\Omega} \rho_m(t, x) \psi_j(x) \cdot \psi_k(x) dx, \tag{4.2}$$

$$\beta_{jkl}^m(t) = \int_{\Omega} \rho_m(t, x) \{ (\psi_k(x) \cdot \nabla) \psi_l(x)) \} \cdot \psi_j(x) dx, \tag{4.3}$$

$$\gamma_j^m(t) = \int_{\Omega} \rho_m(t, x) f_m(t, x) \cdot \psi_j(x) dx \tag{4.4}$$

and $\rho_m(t, x)$ is a solution of (3.4).

The follwing lemma is easily obtained.

Lemma 4.1. *There exists a number* $T_2 \in (0, T_1]$ *such that* (4.1) *has a unique solution* $(a_{m1}(t), \cdots, a_{mm}(t)) \in C^1([0, T_2])^m$.

If we set $v_m(t, x) = \sum_{k=1}^{m} a_{mk}(t)\psi_k(x)$, then we have

Lemma 4.2. *There exist a number* $T' \in (0, T_2]$ *and constant* $c>0$ *which are independent of* m *such that*

$$\|\sqrt{\rho_m} v_{mt}\|_{L^2(0, T'; L^2(\Omega)^3)} + \|v_m\|_{L^\infty(0, T'; H^1_\sigma(\Omega))} + \|v_m\|_{L^2(0, T'; H^2(\Omega)^3)} \leqq c. \tag{4.5}$$

Proof. If we multiply (4.1) by $\frac{d}{dt} a_{mj}(t)$ and sum over $j=1, \cdots, m$, then we get that

$$\int_\Omega \rho_m |v_{mt}|^2 dx + \int_\Omega \rho_m \{(v_m \cdot \nabla) v_m\} \cdot v_{mt} dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v_m|^2 dx = \int_\Omega \rho_m f_m \cdot v_{mt} dx. \tag{4.6}$$

Next multiplying (4.1) by $\lambda_j a_{mj}(t)$ and summing over $j=1, \cdots, m$, then we obtain that

$$\int_\Omega |P\Delta v_m|^2 dx = \int_\Omega \rho_m v_{mt} \cdot P\Delta v_m dx + \int_\Omega \rho_m \{(v_m \cdot \nabla) v_m\} \cdot P\Delta v_m dx - \int_\Omega \rho_m f_m \cdot P\Delta v_m dx. \tag{4.7}$$

By the way, let $u \in H^1_0(\Omega) \cap H^2(\Omega)$, then for any δ with $0 \leqq \delta \leqq 1$ there exists a constant $c_1 > 0$ dependent only on δ and Ω such that

$$\|u\|_{H^{1+\delta}(\Omega)} \leqq c_1 \|\nabla u\|^{1-\delta}_{L^2(\Omega)} \|\Delta u\|^{\delta}_{L^2(\Omega)} \tag{4.8}$$

holds. Hence we find that

$$\begin{aligned} \int_\Omega |v_m|^2 |\nabla v_m|^2 dx &\leqq c_2 \|v_m\|^2_{H^{1+2/3}(\Omega)} \|\nabla v_m\|^2_{L^2(\Omega)} \\ &\leqq c_3 \|\nabla v_m\|^{8/3}_{L^2(\Omega)} \|\Delta v_m\|^{4/3}_{L^2(\Omega)} \\ &\leqq c_4 \|\nabla v_m\|^8_{L^2(\Omega)} + \varepsilon_1 \|P\Delta v_m\|^2_{L^2(\Omega)}, \end{aligned} \tag{4.9}$$

where c_2, c_3 and c_4 are independent of m and ε_1 will be determined later on. Then we have

$$(4.10)\qquad \frac{1}{2}\frac{d}{dt}\|\nabla v_m\|^2_{L^2(\Omega)}+\|\sqrt{\rho_m}v_{mt}\|^2_{L^2(\Omega)}$$
$$\leqq\frac{1}{2}\|\sqrt{\rho_m}v_{mt}\|^2_{L^2(\Omega)}+\varepsilon_1\|P\Delta v_m\|^2_{L^2(\Omega)}+c_5\|\nabla v_m\|^8_{L^2(\Omega)}+c_5\|f_m\|^2_{L^2(\Omega)}$$

and

$$(4.11)\qquad \|P\Delta v_m\|^2_{L^2(\Omega)}\leqq(\frac{3}{4}+\varepsilon_1)\|P\Delta v_m\|^2_{L^2(\Omega)}+c_5\|\sqrt{\rho_m}v_{mt}\|^2_{L^2(\Omega)}$$
$$+c_5\|\nabla v_m\|^8_{L^2(\Omega)}+c_5\|f_m\|^2_{L^2(\Omega)},$$

where c_5 is independent of m.

Let $\varepsilon=1/4c_5$ and $\varepsilon_1=\varepsilon/8(\varepsilon+1)$, then (4.10) + ε (4.11) implies that

$$(4.12)\qquad \frac{1}{2}\frac{d}{dt}\|\nabla v_m\|^2_{L^2(\Omega)}+\frac{1}{4}\|\sqrt{\rho_m}v_{mt}\|^2_{L^2(\Omega)}+\frac{\varepsilon}{8}\|P\Delta v_m\|^2_{L^2(\Omega)}$$
$$\leqq c_6(\|\nabla v_m\|^8_{L^2(\Omega)}+\|f_m\|^2_{L^2(\Omega)}),$$

where c_6 is independent of m.

Therefore we find that there exist a number $T'\in(0,T_2]$ and $c'>0$ which are independent of m such that

$$(4.13)\qquad \|\nabla v_m\|^2_{L^2(\Omega)}\leqq c'\ \text{for all}\ t\in[0,T'].$$

Q.E.D.

§5. Proof of Theorem (third step). In this section we examine the consequences obtained in §§3 and 4.

Let B_R be a closed ball in $C^0([0,T'])^m$ with radius $R\leqq(c'/\lambda_1)^{1/2}$ and let $(\tilde a_{m1}(t),\cdots,\tilde a_{mm}(t))\in B_R$.

On the other hand, by (3.3), (4.13) and Poincaré's lemma, we find that

$$(5.1)\qquad \sum_{k=1}^{m}a^2_{mk}(t)=\|v_m\|^2_{L^2(\Omega)}\leqq\frac{1}{\lambda_1}\|\nabla v_m\|^2_{L^2(\Omega)}\leqq\frac{c'}{\lambda_1}\leqq R^2$$

and consequently

$$(5.2)\qquad (a_{m1}(t),\cdots,a_{mm}(t))\in B_R\cap C^1([0,T'])^m.$$

Therefore we can conclude the map $(\tilde{a}_{m1}(t), \cdots, \tilde{a}_{mm}(t)) \to (a_{m1}(t), \cdots, a_{mm}(t))$ is compact from B_R into itself and it has a fixed point. Namely this fixed point and $\rho_m(t, x)$ defined by (3.6) solve (3.4) and (4.1) and satisfy the estimates obtained in §§3 and 4. Moreover it follows from these estimates that we can extract subsequences, still denoted by ρ_m and v_m, such that

$$\rho_m \to \rho \text{ weak* in } L^\infty([0, T'] \times \Omega) \tag{5.3}$$

and

$$v_m \to v \text{ weak* in } L^\infty(0, T'; H^1_\sigma(\Omega)) \text{ and weakly in } L^2(0, T'; H^2(\Omega)) \tag{5.4}$$

and that

$$\| \rho_{mt} \| L^\infty(0, T'; H^{-1}(\Omega)) \leqq c_7, \tag{5.5}$$

where $c_7 > 0$ is independent of m.

§6. Proof of Theorem (fourth step). Let $\Psi_m(t, x)$ and $\theta_{0m}(x)$ be the regularization of $\frac{\mu}{2} \sum_{j,k=1}^{3} (v^j_{x_k} + v^k_{x_j})^2$ and $\theta_0(x) - \bar{\theta}$ respectively, where v is in (5.4), and consider the following initial-boundary value problem:

$$\begin{cases} c_V \rho_m (\theta_{mt} + \sum_{j=1}^{3} v^j_m \theta_{mx_j}) = \kappa \Delta \theta_m + \Psi_m \\ \theta_m(0, x) = \theta_{0m}(x) \\ \theta_m |_{\partial\Omega} = 0. \end{cases} \tag{6.1}$$

Then this has a unique classical solution $\theta_m(t, x)$ clearly. Moreover we have

Lemma 6.1. *There exists a constant $\tilde{c} > 0$ independent of m such that*

$$\| \sqrt{\rho_m} \theta_m \| L^\infty(0, T'; L^2(\Omega)) + \| \theta_m \| L^2(0, T'; H^1_0(\Omega)) \leqq \tilde{c}. \tag{6.2}$$

Proof. We can easily obtain the equality:

$$\frac{c_V}{2} \int_\Omega \rho_m \theta_m^2 dx + \kappa \int_0^t \int_\Omega |\nabla \theta_m|^2 dx dt \tag{6.3}$$

$$= \frac{c_V}{2}\int_\Omega \rho_{0m}\theta_{0m}^2 dx + \int_0^t \int_\Omega \Psi_m \theta_m dx dt.$$

It follows from Hölder's inequality, Sobolev's imbedding theorem and the properties of mollifier that

$$(6.4) \qquad |\text{right hand side of } (6.3)| \leqq c_8 \| v \|^2_{L^\infty(0,T';H^1_\sigma(\Omega))} \| v \|^2_{L^2(0,T';H^2(\Omega))} + \frac{\kappa}{2} \| \nabla \theta_m \|^2_{L^2(0,T';L^2(\Omega))} + \frac{c_V(M+1)}{2} \| \theta_0 - \bar{\theta} \|^2_{L^2(\Omega)},$$

where $c_8 > 0$ is independent of m.

Therefore the desired estimate is accomplished. Q.E.D.

§7. **Proof of Theorem** (final step). In this section we shall consider the convergence of nonlinear terms as $m \to \infty$. For that purpose we rely on Lions[3].

The consequence that ρ_m is bounded in $L^2([0, T'] \times \Omega)$, v_m is bounded in $L^2(0, T'; H^1_\sigma(\Omega))$ and ρ_{mt} is bounded in $L^2(0, T'; H^{-1}(\Omega))$ and the compensated compactness imply that $\rho_m v_m \to \rho v$ and $\rho_m \theta_m \to \rho\theta$ in the sense of distributions.

Next it follows from estimates obtained in §§3, 4 and 5 that $\rho_m v^j_m$ is bounded in $L^2(0, T'; L^6(\Omega))$ and $(\rho_m v^j_m)_t$ is bounded in $L^2(0, T'; H^{-1}(\Omega))$, $j=1, 2, 3$. Thus we can extract a subsequence, still denoted by $\rho_m v^j_m$, such that $\rho_m v^j_m \to \xi$ weakly in $L^2(0, T'; L^6(\Omega))$. Therefore the compensated compactness implies that $\rho_m v^j_m v_m \to \xi v$ and $\rho_m v^j_m \theta_m \to \xi\theta$, $j=1, 2, 3$, in the sense of distributions. But we have already shown that $\xi = \rho v^j$. So we find that the limit functions $\rho(t, x)$, $v(t, x)$ and $\theta(t, x)$ satisfy the conditions in Definition.

This completes the proof of Theorem.

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