Uniqueness in the Cauchy Problem for Some System of Nonlinear Equations

ある非線形方程式系に対するコーシー問題における一意性

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Abstract. We give a sufficient condition on the uniqueness in the Cauchy problem for a system of nonlinear equations related to one dimensional motion of viscous isentropic gas.

§1. Introduction and Result

The purpose of the present paper is to show the uniqueness in the Cauchy problem for the system

$$(1.1)\qquad \begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x = k(v)_{tx}, \; x \in \mathrm{R}, \; t > 0 \end{cases}$$

under suitable assumptions.

We can easily find that (1.1) is formally rewritten to the system

$$(1.2)\qquad \begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x = (k'(v)u_x)_x. \end{cases}$$

This is the system of the Navier-Stokes equations of compressible, isentropic flow in Lagrangean coordinates. Here v, u and p are the specific volume, velocity and pressure in the fluid, and one usually takes $p(v) = v^{-\gamma}(\gamma > 1)$ and $k'(v) = v^{-1}$ in gas dynamics.

By the way, in [1] Hoff succeeded to prove the existence of global weak solutions for the system (1.2) with discontinuous initial data $(v_0, u_0)(x)$ satisfying $c^{-1} \leqq v_0(x) \leqq c$ for some positve constant c, $v_0 - v' \in \mathrm{L}^2 \cap \mathrm{BV}$ for some fixed $v' > 0$ and $u_0 \in \mathrm{L}^2$; however, he did not prove the uniqueness of solutions.

Generally speaking, in nonlinear problem uniqueness theorems cannot compare with existence theorems for number. This is more striking in the case of nonsmooth initial data.

Now let us define a weak solution of the Cauchy problem for the system (1.1).

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Definition. We say that $(v, u)(t, x)$ is a weak solution of the Cauchy problem for the system (1.1) if for any C^{∞}-functions φ and ψ with compact support in x such that $\varphi(T, x)=\psi(T, x)=0$,

$$(1.3)\qquad \int_0^T\int_R (v\varphi_t - u\varphi_x)\,dt\,dx + \int_R v_0(x)\varphi(0, x)\,dx = 0$$

and

$$(1.4)\qquad \int_0^T\int_R (u\psi_t + p(v)\psi_x + k(v)\psi_{tx})\,dt\,dx + \int_R (u_0(x)\psi(0, x) - k(v_0)\psi_x(0, x))\,dx = 0.$$

hold.

Next we explain the basic idea in [1]. A result of Hoff-Smoller [2] shows that, due to the parabolicity in the second equation in (1.2), discontinuities in $u_0(x)$ must be smoothed out $t>0$, but that discontinuities in $v_0(x)$ persist for all t. Then Rankine-Hugoniot condition $s[v]=-[u]$ ($[v]$ (resp. $[u]$) denotes the jump in v (resp. u) across the discontinuity and s the speed of the discontinuity) implies that $s=0$. That is, we expect that discontinuities in v propagates along the particle path $x=constant$ and that u is continuous in $t>0$. Though we cannot go into details here, a solution obtained in [1] has the properties that $\tilde{c}^{-1}\leqq v(t, x)\leqq \tilde{c}$ for some positive constant $\tilde{c}$ and $v(t, x)$ is piecewise Hölder continuous, where we use Chapter 2 Lemma 3.1 in [3] to get this.

Under the above observation, we have

Theorem. *Suppose that*

(1.5) $p(v),\ k(v)\in C^2(v>0)$

and

(1.6) *there exists a constant* $\delta>0$ *such that* $k'(v)\geqq\delta$.

If weak solutions of the Cauchy problem for (1.1) *satisfy the conditions that* $v(t, x)$ *is piecewise Hölder continuous and* $u(t, x)$ *is continuous, then the uniqueness holds almost everywhere in* $t>0$.

§2. Proof of Theorem

Let (v_1, u_1) and (v_2, u_2) be two solutions with the same initial data and set

$$(2.1)\qquad a(t, x)=\int_0^1 p'(sv_1+(1-s)v_2)\,ds$$

and

$$b(t, x) = \int_0^1 k'(sv_1 + (1-s)v_2)\,ds. \tag{2.2}$$

Then by (1.3) and (1.4) we find that

$$\int_0^T \int_R \{(u_1 - u_2)(\psi_t - \varphi_x) + (v_1 - v_2)(\varphi_t + a\psi_x + b\psi_{tx})\}\,dt\,dx = 0. \tag{2.3}$$

Now we assume that v_0 is discontinuous at $x = x_1, \cdots, x_m$,

For $j = 0, \cdots, m$, we set $\Omega_j = (x_j, x_{j+1})$ and $Q_{T,j} = (0, T] \times \Omega_j$, where $x_0 = -\infty$ and $x_{m+1} = \infty$.

For any $f, g \in C_0^\infty([0, T] \times R)$, we set

$$f_j(t, x) = \begin{cases} f & \text{in } \bar{Q}_{T,j} \\ 0 & \text{otherwise}, \quad j = 0, \cdots, m \end{cases}$$

and

$$g_j(t, x) = \begin{cases} g & \text{in } \bar{Q}_{T,j} \\ 0 & \text{otherwise}, \quad j = 0, \cdots, m \end{cases}$$

and consider the following boundary value problem.

$$\begin{cases} \psi_{jt} - \varphi_{jx} = f_j \\ \varphi_{jt} + a\psi_{jx} + b\psi_{jtx} = g_j & \text{in } Q_{T,j} \\ \varphi_j(T, x) = \psi_j(T, x) = 0 & \text{on } \Omega_j \\ \varphi_j|_{\partial\Omega_j} = 0 & t \in [0, T]. \end{cases} \tag{2.4}$$

Lemma 2.1. *For every j with $j = 0, \cdots, m$, the system (2.4) possesses solutions φ_j and ψ_j such that φ_j, φ_{jt}, φ_{jx} ψ_j, ψ_{jt}, ψ_{jx} and ψ_{jtx} are bounded and continuous in $\bar{Q}_{T,j}$.*

Proof. Using the first equation in (2.4), we can rewrite the second equation in (2.4) to

$$\varphi_{jt} + a\psi_{jx} + b\varphi_{jxx} = g_j - bf_{jx}.$$

We prove this lemma by the method of successive approximation and consider the following scheme.

$$(2.5)_j^0 \quad \begin{cases} \psi_{jt}^0 = f_j \\ \varphi_{jt}^0 + a\psi_{jx}^0 + b\varphi_{jxx}^0 = g_j - bf_{jx} \\ \varphi_j^0(T,\ x) = \psi_j^0(T,\ x) = 0 \\ \varphi_j^0|_{\partial\Omega_j} = 0 \end{cases}$$

and for $i \geqq 1$,

$$(2.5)_j^i \quad \begin{cases} \psi_{jt}^i = \varphi_{jx}^{i-1} \\ \varphi_{jt}^i + a\psi_{jx}^i + b\varphi_{jxx}^i = 0 \\ \varphi_j^i(T,\ x) = \psi_j^i(T,\ x) = 0 \\ \varphi_j^i|_{\partial\Omega_j} = 0. \end{cases}$$

We note that a and b are Hölder continuous in $\bar{Q}_{T,j}$ with the exponent α_j, where α_j is Hölder exponent of v in $\bar{Q}_{T,j}$, and (1.6).
Then, inductively we find that for $j = 0, \cdots, m$, $r = 0, 1$ and $s = 0, 1, 2$,

$$|\partial_x^r \psi_j^i|,\ |\partial_x^s \varphi_j^i| \leqq const.\{(T-t)^{i-\alpha_j/2}/i\,!\}$$

and since $f,\ g \in C_0^\infty([0,\ T] \times R)$, for $j = 0,\ m$, $r = 0, 1$ and $s = 0, 1, 2$

$$|\partial_x^r \psi_j^i|,\ |\partial_x^s \varphi_j^i| \leqq const.\{(T-t)^{i+\alpha_j/2}/i\,!\}exp(-\hat{c}|x|^2)$$

as $|x| \to \infty$, where $\hat{c}$ is a positive constant. Let

$$\psi_j = \sum_{i=0}^{\infty} \psi_j^i \quad \text{and} \quad \varphi_j = \sum_{i=0}^{\infty} \varphi_j^i,$$

then we have the assertion of this lemma. Q.E.D.

Lemma 2.2.. *There exists a positive constant M such that $|\varphi_0| \leqq Me^x$ in $Q_{T,0}$ and $|\varphi_m| \leqq Me^{-x}$ in $Q_{T,m}$*

Proof.

$$L\varphi_m \equiv \varphi_{mt} + b\varphi_{mxx} = g_m - bf_{mx} - a\psi_{mx} \equiv G_m.$$

We set

$$w_m^{\pm} = M_1 exp(-x + A(T-t)) \pm \varphi_m,$$

where M_1 and A are sufficiently large constants. Then

$$Lw_m^{\pm} = M_1(-A+b) exp(-x+A(T-t)) \pm G_m \leqq 0,$$

if we refer to the estimate in the proof of Lemma 2.1. Moreover

$$w_m^{\pm}(T, x) = M_1 e^{-x} \geqq 0.$$

and

$$w_m^{\pm}(t, x_m) = M_1 exp(-x_m + A(T-t)) \pm \varphi_m(t, x_m) \geqq 0.$$

Hence by the comparison theorem, we find that $w_m^{\pm} \geqq 0$. Therefore

$$|\varphi_m| \leqq M_1 exp(-x+A(T-t)) \leqq Me^{-x}.$$

The estimate for φ_0 is similar. Q.E.D.

Lemma 2.3. *φ_{0x} and φ_{mx} are in L^2 with respect to x.*

Proof. If we apply Lemma 2.1 and Lemma 2.2 to

$$\int_{-\infty}^{x_1} (\varphi_{0x})^2 dx \ \ and \ \int_{x_m}^{\infty} (\varphi_{mx})^2 dx,$$

then the assertion is easily seen by the direct calculation. Q.E.D.

Remark. From the first equation in (2.4) and Lemma 2.3, we have that ψ_0, ψ_{0t}, ψ_m and ψ_{mt} are in L^2 with respect to x.

Now we put

$$\tilde{\varphi}_j = \begin{cases} \varphi_j & in\ \bar{Q}_{T,j} \\ 0 & otherwise \end{cases}$$

and

$$\tilde{\varphi}=\sum_{j=0}^{m}\tilde{\varphi}_j.$$

$\tilde{\psi}$ is defined by the same method.

Proof of Theorem. Let $\chi_N(x)$ be a C^∞-function satisfying the following property :

$$0\leqq\chi_N(x)\leqq 1 \tag{2.6}$$

$$\chi_N(x)=\begin{cases}1 & for\ |x|\leqq N\\ 0 & for\ |x|\geqq N+1\end{cases} \tag{2.7}$$

and let $\tilde{\varphi}^h$(resp.$\tilde{\psi}^h$) be the regularization of $\tilde{\varphi}$ (resp. $\tilde{\psi}$).

Substituting $\chi_N\tilde{\varphi}^h$ and $\chi_N\tilde{\psi}^h$ for (1.3) and (1.4) respectively, then we have

$$\begin{aligned}
0&=\int_0^T\int_R[(u_1-u_2)\{(\chi_N\tilde{\psi}^h)_t-(\chi_N\tilde{\varphi}^h)_x\}\\
&\qquad+(v_1-v_2)\{(\chi_N\tilde{\varphi}^h)_t+a(\chi_N\tilde{\psi}^h)_x+b(\chi_N\tilde{\psi}^h)_{tx}\}]dtdx\\
&=\int_0^T\int_R[(u_1-u_2)\chi_N(\tilde{\psi}_t-\tilde{\varphi}_x)+(v_1-v_2)\chi_N(\tilde{\varphi}_t+a\tilde{\psi}_x+b\tilde{\psi}_{tx})]dtdx\\
&\quad+\int_0^T\int_R[(u_1-u_2)\chi_N\{(\tilde{\psi}^h_t-\tilde{\psi}_t)-(\tilde{\varphi}^h_x-\tilde{\varphi}_x)\}\\
&\quad+(v_1-v_2)\chi_N\{(\tilde{\varphi}^h_t-\tilde{\varphi}_t)+a(\tilde{\psi}^h_x-\psi_x)+b(\tilde{\psi}^h_{tx}-\tilde{\psi}_{tx})\}]dtdx\\
&\quad+\int_0^T\int_R\chi_{Nx}\{(u_1-u_2)(-\tilde{\varphi}^h)+(v_1-v_2)(a\tilde{\psi}^h+b\tilde{\psi}^h_t)\}dtdx\\
&\equiv I_1+I_2+I_3.
\end{aligned}$$

From Lemma 2.1 and the definition of $\tilde{\varphi}$ and $\tilde{\psi}$, we get that for sufficiently large N,

$$\begin{aligned}
I_1&=\int_0^T\int_R\{(u_1-u_2)\chi_N f+(v_1-v_2)\chi_N g\}dtdx\\
&=\int_0^T\int_R\{(u_1-u_2)f+(v_1-v_2)g\}dtdx.
\end{aligned}$$

Next for any $\varepsilon>0$ and a fixed N, $|I_2|<\varepsilon/2$ as $h\to 0$.

Moreover we note that $\chi_{Nx}=0$ for $|x|\leqq N$ and $|x|\geqq N+1$. From Lemma 2.2, Lemma 2.3

and Remark, we obtain that for any $\varepsilon > 0$ and a sufficiently large N, $|I_3| < \varepsilon/2$.

Therefore we find that for any $\varepsilon > 0$,

$$\left| \int_0^T \int_R \{ (u_1 - u_2) f + (v_1 - v_2) g \} \mathrm{d}t \mathrm{d}x \right| < \varepsilon.$$

Hence we have the conclusion by the arbitrariness of f and g.
This completes the proof.

References

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