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On the Euler Equations of a Nonhomogeneous Ideal Incompressible Fluid

ノンホモジニアウスな非圧縮性理想流体のオイラー方程式について

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Abstract.

It is shown here that the Cauchy problem for the Euler equations of a nonhomogeneous ideal incompressible fluid has a unique solution for a small time interval. The existence of a solution is established by applying the method of the semi Galerkin approximations.

§1. Introduction.

Consider the system of equations

$$\begin{cases} \rho_t + v\cdot\nabla\rho = 0 \\ \rho[v_t + (v\cdot\nabla)v] + \nabla p = \rho f \\ div\, v = 0 \end{cases} \tag{1.1}$$

in $Q_T = \mathbb{R}^3 \times [0,T]$, subject to the initial conditions

$$\begin{cases} \rho|_{t=0} = \rho_0(x) \\ v|_{t=0} = v_0(x). \end{cases} \tag{1.2}$$

Here $f(x,t)$, $\rho_0(x)$ and $v_0(x)$ are given, while the density $\rho(x,t)$, the velocity vector $v(x, t) = (v^1(x,t), v^2(x,t), v^3(x,t))$ and the pressure $p(x,t)$ are unknowns. The system (1.1) describes the motion of a nonhomogeneous ideal incompressible fluid.

Our theorem is the following.

Theorem. *Assume that*

$$\rho_0(x) - \bar\rho \in H^3(\mathbb{R}^3) \text{ for some positive constant } \bar\rho, \tag{1.3}$$

$$\inf \rho_0(x) \equiv m > 0 \text{ and } \sup \rho_0(x) \equiv M < \infty, \tag{1.4}$$

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$$v_0(x) \in \mathrm{H}^3(\mathbb{R}^3) \ \textit{and} \ \mathit{div}\ v_0 = 0 \tag{1.5}$$

and

$$f(x, t) \in \mathrm{L}^2(0, \mathrm{T}: \mathrm{H}^3(\mathbb{R}^3)). \tag{1.6}$$

Then there exists $\mathrm{T}^* \in (0, \mathrm{T}]$ *such that the problem* (1.1) *and* (1.2) *has a unique solution* (ρ, v, p) *which satisfies*

$$(\rho - \bar{\rho},\ v,\ \nabla p) \in \mathrm{L}^\infty(0, \mathrm{T}^*: \mathrm{H}^3(\mathbb{R}^3)) \times \mathrm{L}^\infty(0, \mathrm{T}^*: \mathrm{H}^3(\mathbb{R}^3)) \times \mathrm{L}^2(0, \mathrm{T}^*: \mathrm{H}^3(\mathbb{R}^3)). \tag{1.7}$$

§2. Preliminaries.

In this section we shall obtain an a priori estimate for solutions of (1.1) and (1.2). Let (ρ, v, p) be a sufficiently regular solution. We assume, for simplicity, $f = 0$. The general case can be treated in the same way.

Lemma 2.1. *If we put* $\tilde{\rho} = \rho - \tilde{\rho}$, *then*

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\tilde{\rho}(t)\|_3 \leqq c_1 \|v(t)\|_3 \|\tilde{\rho}(t)\|_3, \tag{2.1}$$

where c_1 *is a positive constant depending only on imbedding theorems and* $\|\cdot\|_{\mathrm{K}} = \|\cdot\|_{\mathrm{H}^{\mathrm{K}}(\mathbb{R}^3)}$.

Proof. It follows $(1.1)_1$ and $(1.2)_1$ that $\tilde{\rho}$ satisfies the equation

$$\begin{cases} \tilde{\rho}_t + v \cdot \nabla \tilde{\rho} = 0 \\ \tilde{\rho}|_{t=0} = \tilde{\rho}_0(x). \end{cases} \tag{2.2}$$

Applying the operator D^α $(= (\partial/\partial x_1)^{\alpha_1} (\partial/\partial x_2)^{\alpha_2} (\partial/\partial x_3)^{\alpha_3})$ to $(2.2)_1$, multiplying the result by $D^\alpha \tilde{\rho}$, integrating over $\mathbb{R}^3$ and adding in α with $|\alpha|$ $(= \alpha_1 + \alpha_2 + \alpha_3) \leqq 3$, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\tilde{\rho}(t)\|_3^2 \leqq \mathrm{c}_2 \|v(t)\|_3 \|\tilde{\rho}(t)\|_3^2. \tag{2.3}$$

Hence it is easy to see that (2.1) holds.

Q.E.D.

Lemma 2.2. *Put*

$$\Phi(t)=\sum_{j=0}^{3}\sum_{|\alpha|=j}\|\sqrt{\rho(t)}\,D^{\alpha}v(t)\|_0. \tag{2.4}$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t)\leqq c_3[1+\|\tilde{\rho}(t)\|_3+\Phi(t)]^5, \tag{2.5}$$

where c_3 *is a positive constant depending only on* m, M *and imbedding theorems.*

Proof. We first note that

$$\mathrm{m}\leqq\rho(x,t)\leqq\mathrm{M}, \tag{2.6}$$

since we have the representation

$$\rho(x,t)=\rho_0(y(\tau,x,t)|_{\tau=0}), \tag{2.7}$$

where $\mathrm{y}(\tau,x,t)$ is the solution of the Cauchy problem

$$\begin{cases}\dfrac{\mathrm{d}y}{\mathrm{d}\tau}=v(y,\tau)\\ y|_{\tau=t}=x.\end{cases} \tag{2.8}$$

Therefore we find that

$$\sqrt{\mathrm{m}}\,\|v\|_3\leqq\Phi(t)\leqq\sqrt{\mathrm{M}}\,\|v\|_3. \tag{2.9}$$

(i) We multiply $(1.1)_2$ by v and integrate over $\mathbb{R}^3$. Taking account of $(1.1)_1$ and $(1.1)_3$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}\,v\|_0=0. \tag{2.10}$$

Multiplying by v_t and integrating over $\mathbb{R}^3$, then

$$m\|v_t\|_0^2 \leqq M\|v\|_1 \; \|Dv\|_1 \; \|v_t\|_0 \;, \tag{2.11}$$

where we use the notation $D^k u = \sum_{|\alpha|=k} D^\alpha u$. Thus

$$\|v_t\|_0 \leqq c_4 \|v\|_2^2 \,. \tag{2.12}$$

(ii) Apply the operator D^α with $|\alpha|=1$ on each side of $(1.1)_2$, multiply the result by $D^\alpha v$ and integrate over $\mathbb{R}^3$. Noting (2.9) and (2.12), then, similarly to (i), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}\,Dv\|_0^2 &\leqq \|D\rho\|_2 \; \|v_t\|_0 \; \|Dv\|_0 + \|Dv\|_2 \; \|\sqrt{\rho}\,Dv\|_0^2 + \|D\rho\|_2 \; \|v\|_2 \; \|Dv\|_0^2 \\ &\leqq c_5(\|D\rho\|_2 \; \|v\|_2^2 \|\sqrt{\rho}\,Dv\|_0^2 + \|Dv\|_2 \; \|\sqrt{\rho}\,Dv\|_0^2 + \|D\rho\|_2 \; \|v\|_2 \; \|\sqrt{\rho}\,Dv\|_0^2\,).\end{aligned} \tag{2.13}$$

Hence we have

$$\frac{d}{dt}\|\sqrt{\rho}\,Dv\|_0 \leqq c_6(1+\|D\rho\|_2 + \|v\|_3)^3. \tag{2.14}$$

If we multiply by $D^\alpha v_t$ and integrate over $\mathbb{R}^3$, then we obtain

$$\begin{aligned}m\|Dv_t\|_0^2 \leqq c_7(&\|D\rho\|_2 \; \|v_t\|_0 \; \|Dv_t\|_0 + \|v\|_2 \; \|D^2 v\|_0 \; \|Dv_t\|_0 \\ &+ \|Dv\|_1^2 \|Dv_t\|_0 + \|D\rho\|_2 \; \|v\|_2 \; \|Dv\|_0 \; \|Dv_t\|_0\,).\end{aligned} \tag{2.15}$$

Therefore

$$\|Dv_t\|_0 \leqq c_8(1+\|D\rho\|_2)\,\|v\|_2^2. \tag{2.16}$$

(iii) Making use of the operator D^α with $|\alpha|=2$ in place of the operator D^α with $|\alpha|=1$ and repeating the argument in (ii), we have

$$\frac{d}{dt}\|\sqrt{\rho}\,D^2 v\|_0 \leqq c_9(1+\|D\rho\|_2 + \|v\|_3)^4 \tag{2.17}$$

and

$$\|D^2 v_t\|_0 \leqq c_{10}(1+\|D\rho\|_2)^2\|v\|_3^2. \tag{2.18}$$

(iv) Apply the operator D^α with $|\alpha|=3$ to $(1.1)_2$, multiply by $D^\alpha v$ and integrate over $\mathbb{R}^3$. Then, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}\, D^3 v\|_0 \leqq c_{11}(1+\|D\rho\|_2+\|v\|_3)^5. \tag{2.19}$$

Consequently, it follows from (2.10), (2.14), (2.17) and (2.19) that (2.5) holds.

Q.E.D.

Proposition 2.3. *There exists* $\mathrm{T}^* \in (0,\mathrm{T}]$ *such that*

$$\|\tilde{\rho}(t)\|_3 + \|v(t)\|_3 \leqq c \qquad \textit{for } t \leqq \mathrm{T}^*, \tag{2.20}$$

where c *is a positve constant depending only on* m, M, $\|\tilde{\rho}_0\|_3$, $\|v_0\|_3$ *and imbedding theorems.*

Proof. If we set

$$\Psi(t) = 1 + \|\tilde{\rho}(t)\|_3 + \Phi(t), \tag{2.21}$$

then, from Lemma 2.1 and Lemma 2.2, we have a differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(\mathrm{t}) \leqq \tilde{c}\Psi(\mathrm{t})^5, \tag{2.22}$$

where $\tilde{c}=c_1+c_3$. Thus we conclude that

$$\Psi(t) \leqq \Psi(0)\,(1-4\tilde{c}t\Psi(0)^4)^{-1/4} \qquad \textit{as long as } t < (4\tilde{c}\Psi(0)^4)^{-1}. \tag{2.23}$$

Q.E.D.

§3. Proof of Theorem.

We solve the problem (1.1) and (1.2) by applying the semi Galerkin method with the basis $\{\varphi_k(x)\}$ in $\mathrm{H}^4(\mathbb{R}^3) \cap \mathrm{J}$, where $\mathrm{J}=\{u \in \{\mathrm{C}_0^\infty(\mathbb{R}^3)\}^3 : div\ u=0\}$. Let us look for $\rho_n(x,t)$ and

$$v_n(x,t)=\sum_{j=1}^{n} a_{nj}(t)\,\varphi_j(x) \tag{3.1}$$

satisfying

$$\begin{cases} \rho_{nt}+v_n\cdot\nabla\rho_n=0 \\ ((\rho_n[v_{nt}+(v_n\cdot\nabla)v_n],\ \varphi_k))=0, \quad k=1,\cdots,m \\ \rho_n|_{t=0}=\rho_0(x) \\ v_n|_{t=0}=\sum_{j=1}^{n} a_j\varphi_j(x), \qquad a_j=\int_{\mathbb{R}^3} v_0\cdot\varphi_j\,dx, \end{cases} \tag{3.2}$$

where $((\cdot\ ,\ \cdot))$ stands for the scalar product in $H^3(\mathbb{R}^3)$.

If we multiply $(3.2)_2$ by $a_{nk}(t)$ and add in k, then we obtain

$$((\rho_n[v_{nt}+(v_n\cdot\nabla)v_n],\ v_n))=0. \tag{3.3}$$

Therefore, similar to §2, a priori estimates

$$\mathrm{m}\leqq\rho_n(x,t)\leqq\mathrm{M} \tag{3.4}$$

and

$$\|\tilde{\rho}_n(t)\|_3+\|v_n(t)\|_3\leqq c \qquad for\ \ t\leqq\mathrm{T}^* \tag{3.5}$$

hold. These estimates guarantee the unique solvability of the problem (3.2), and, furthermore, permit to pass to the limit using the standard compactness arguments (cf. [1], [2], [3]).

Hence we can verify the existence of a unique solution of the problem (1.1) and (1.2) as well as the applicability of the inequalities (2.6) and (2.20).

This completes the proof.

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