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ON THE VANISHING VISCOSITY IN THE CAUCHY PROBLEM FOR EQUATIONS OF A NONHOMOGENEOUS INCOMPRESSIBLE FLUID II

ノンホモジニアウスな非圧縮性流体の方程式に対するコーシー問題における粘性消滅についてII

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ABSTRACT We investigate the Cauchy problem for Euler and Navier-Stokes equations of a nonhomogeneous incompressible fluid in $\mathbb{R}^3$. The unique solvability on a small time interval independent of viscosity is proved, and moreover, it is shown that the solution of Navier-Stokes equations converges in some Hilbert space to the one of Euler equations as viscosity tends to zero.

Key words : Incompressible fluid, Navier-Stokes equations, Euler equations, Vanishing viscosity

1. INTRODUCTION

We consider the system of equations

$$(1.1)\qquad \begin{cases} \rho_t + v \cdot \nabla \rho = 0, \\ \rho[v_t + (v \cdot \nabla) v] + \nabla p = \mu \triangle v + \rho f, \\ \operatorname{div} v = 0, \end{cases}$$

in $Q_T = \mathbb{R}^3 \times [0,T]$, $T>0$, subject to the initial conditions

$$(1.2)\qquad \begin{cases} \rho|_{t=0} = \rho_0(x), \\ v|_{t=0} = v_0(x). \end{cases}$$

Here $f(x,t)$ is a given vector field of external forces, while the density $\rho(x,t)$, the velocity vector $v(x,t)$ and the pressure $p(x,t)$ are the unknowns. The viscosity coefficient μ is

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assumed to be a nonnegative constant.

In these equations, $p(x,t)$ is automatically determined (up to a function of t) by $\rho(x,t)$ and $v(x,t)$, namely, by solving the equation

$$(1.3) \qquad \operatorname{div}(\rho^{-1}\nabla p)=\operatorname{div}(f-(v\cdot\nabla)v+\mu\rho^{-1}\triangle v).$$

Thus we mention (ρ,v) when we talk about the solution of problem (1.1), (1.2).

Compared with the previous paper [2], in which the similar results were proved, we discuss the problem under the weaker assumptions to given data.

The purpose of this paper is to prove

Theorem. *Let* $0\leq\mu\leq 1$, *and assume that*

$$(1.4) \qquad \rho_0(x)\in C^0(\mathbb{R}^3),\ \ \nabla\rho_0(x)\in H^2(\mathbb{R}^3),\ \ 0<m\leq\rho_0(x)\leq M<\infty,$$

$$(1.5) \qquad v_0(x)\in H^3(\mathbb{R}^3),\ \ \operatorname{div} v_0=0,$$

$$(1.6) \qquad f(x,t)\in L^2(0,T;H^3(\mathbb{R}^3)).$$

Then there exists $T_0\in(0,T]$ *independent of* μ *such that problem* (1.1), (1.2) *has a unique solution* $(\rho,v)(x,t)$ *which satisfies*

(1.7)

$$\rho(x,t)\in C^0(\mathbb{R}^3\times[0,T_0]),\ \nabla\rho(x,t)\in C^0([0,T_0];\ H^2(\mathbb{R}^3)),\ 0<m\leq\rho(x,t)\leq M<\infty,$$

$$(1.8) \qquad v(x,t)\in C^0([0,T_0];\ H^3(\mathbb{R}^3)).$$

Furthrmore, let $(\rho^0,\ v^0)$ *be the solution of problem* (1.1), (1.2) *with* $\mu=0$ *and* $(\rho^\mu,\ v^\mu)$ *the one with* $\mu>0$, *then we have*

$$(1.9) \qquad \sup_{0\leq t\leq T_0}[\|(\rho^0-\rho^\mu)(t)\|_2+\|(v^0-v^\mu)(t)\|_2]\to 0 \text{ as } \mu\to 0,$$

where $\|\cdot\|_k=\|\cdot\|_{H^k(\mathbb{R}^3)}$.

2. Preliminaries

In this section we establish several *a priori* estimates for solutions of problem (1.1), (1.2). Let $(\rho,v)(x,t)$ be a sufficiently regular solution. Hereafter C stands for the generic constant independent of μ.

Lemma 2.1. *Let*

$$\Psi(t)=\int_0^t [1+\|\nabla\rho(s)\|_2^2+\|v(s)\|_3^2]^2 ds, \tag{2.1}$$

then the estimates

$$m\leq\rho(x,t)\leq M \tag{2.2}$$

and

$$\|\nabla\rho(t)\|_2^2\leq\|\nabla\rho_0\|_2^2+C\Psi(t) \tag{2.3}$$

hold.

Proof. It is well-known that, according to the classical method of characteristics, the solution of problem $(1.1)_1$, $(1.2)_1$ is given by

$$\rho(x,t)=\rho_0(y(\tau,x,t)|_{\tau=0}), \tag{2.4}$$

where $y(\tau,x,t)$ is the solution of the Cauchy problem

$$\begin{cases} \dfrac{dy}{d\tau}=v(y,\tau), \\ y|_{\tau=t}=x. \end{cases} \tag{2.5}$$

From this, the estimate (2.2) results.

Next let us establish (2.3). Apply the operator D_x^α on each side of $(1.1)_1$. Multiplying the result by $D_x^\alpha\rho$, integrating over $\mathbb{R}^3$ and summing over $|\alpha|=1, 2, 3$, we have the equality

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla\rho(t)\|_2^2 &= -\sum_{|\alpha|=1}^{3}\int_{\mathbb{R}^3}(v\cdot\nabla D_x^\alpha\rho)D_x^\alpha\rho dx \\ &\quad -\sum_{|\alpha|=1}^{3}\sum_{0<\beta\leq\alpha}\binom{\alpha}{\beta}\int_{\mathbb{R}^3}(D_x^\beta v\cdot\nabla D_x^{\alpha-\beta}\rho)D_x^\alpha\rho dx. \end{aligned} \tag{2.6}$$

The first term of the right hand side is zero, by integration by parts, since div $v=0$. The second term can be estimated as follows:

$$\sum_{|\alpha|=1}^{3}\sum_{0<\beta\leq\alpha}\binom{\alpha}{\beta}\left|\int_{\mathbb{R}^3}(D_x^{\beta}v\cdot\nabla D_x^{\alpha-\beta}\rho)D_x^{\alpha}\rho dx\right|\leq C\|\nabla v(t)\|_2\|\nabla\rho(t)\|_2^2.$$

Hence we get

$$\frac{d}{dt}\|\nabla\rho(t)\|_2^2\leq C\|\nabla v(t)\|_2\|\nabla\rho(t)\|_2^2, \tag{2.7}$$

and thus (2.3) is obtained. □

Lemma 2.2. *If we put*

$$A=1+\|\nabla\rho_0\|_2^2+\|v_0\|_3^2+\int_0^T\|f(t)\|_3^2dt, \tag{2.8}$$

then we have the inequality

$$\|v(t)\|_3^2+\int_0^t\|v_t(s)\|_2^2ds+\mu\int_0^t\|\nabla v(s)\|_3^2ds\leq C[A+\Psi(t)]^3. \tag{2.9}$$

Proof. By applying the operator D_x^{α} on both sides of $(1.1)_2$, we obtain the equation

$$\begin{aligned}\rho[D_x^{\alpha}v_t+(v\cdot\nabla)D_x^{\alpha}v]+\nabla D_x^{\alpha}p &= \mu\triangle D_x^{\alpha}v-\sum_{0<\beta\leq\alpha}\binom{\alpha}{\beta}D_x^{\beta}\rho D_x^{\alpha-\beta}v_t \\ &\quad -\sum_{0<\beta\leq\alpha}\binom{\alpha}{\beta}\sum_{0\leq\gamma\leq\beta}\binom{\beta}{\gamma}D_x^{\gamma}\rho(D_x^{\beta-\gamma}v\cdot\nabla)D_x^{\alpha-\beta}v \\ &\quad +\sum_{0\leq\beta\leq\alpha}\binom{\alpha}{\beta}D_x^{\beta}\rho D_x^{\alpha-\beta}f.\end{aligned} \tag{2.10}$$

Step 1. Multiplying (2.10) by $D_x^{\alpha}v$ and integrating over $\mathbb{R}^3$, then by making use of

$(1.1)_{1,3}$, (2.2), we have the inequalities

(2.11;0)
$$\frac{1}{2}\frac{d}{dt}\| \sqrt{\rho} v \|_0^2+\mu\|\nabla v\|_0^2\leq C\|f\|_0\|v\|_0$$

and for $k=1,2,3,$

(2.11;k)
$$\frac{1}{2}\frac{d}{dt}\| \sqrt{\rho}\nabla^k v\|_0^2+\mu\|\nabla^{k+1}v\|_0^2$$

$$\leq C[\|\nabla\rho\|_2\|v_t\|_{k-1}\|v\|_3+(1+\|\nabla\rho\|_2)(\|f\|_3+\|v\|_3^2)\|v\|_3]$$

$$\leq C[\|\nabla\rho\|_2^2\|v\|_3^2+(1+\|\nabla\rho\|_2)(\|f\|_3+\|v\|_3^2)\|v\|_3]+\frac{m}{2}\|v_t\|_{k-1}^2.$$

Step 2. If we multiply (2.10) by $D_x^\alpha v_t$ and integrate over $\mathbb{R}^3$, then we obtain

(2.12;0)
$$m\|v_t\|_0^2+\frac{\mu}{2}\frac{d}{dt}\|\nabla v\|_0^2\leq C[\|v\|_2^4+\|f\|_0^2]+\frac{m}{2}\|v_t\|_0^2$$

and for k=1, 2,

(2.12;k)
$$m\|\nabla^k v_t\|_0^2+\frac{\mu}{2}\frac{d}{dt}\|\nabla^{k+1}v\|_0^2$$

$$\leq C[\|\nabla\rho\|_2^2\|v_t\|_{k-1}^2+(1+\|\nabla\rho\|_2)^2(\|f\|_2^2+\|v\|_3^2)]+\frac{m}{2}\|\nabla^k v_t\|_0^2,$$

which mean

(2.13;0)
$$m\|v_t\|_0^2+\mu\frac{d}{dt}\|\nabla v\|_0^2\leq C[\|v\|_2^4+\|f\|_0^2]$$

and for k=1, 2,

(2.13;k)
$$m\|\nabla^k v_t\|_0^2+\mu\frac{d}{dt}\|\nabla^{k+1}v\|_0^2$$

$$\leq C[\|\nabla\rho\|_2^2\|v_t\|_{k-1}^2+(1+\|\nabla\rho\|_2)^2(\|f\|_2^2+\|v\|_3^2)].$$

Step 3. Adding (2.11;1) to (2.13;0), we get

$$\frac{d}{dt}\|\sqrt{\rho}\,\nabla v\|_0^2+2\mu\|\nabla^2 v\|_0^2+m\|v_t\|_0^2+2\mu\frac{d}{dt}\|\nabla v\|_0^2 \tag{2.14}$$

$$\leq C[(1+\|\nabla\rho\|_2^2+\|v\|_3^2)^2+\|f\|_3^2],$$

and noting that $0\leq\mu\leq 1$,

$$\|\nabla v\|_0^2+\int_0^t\|v_t\|_0^2 ds+\mu\int_0^t\|\nabla^2 v\|_0^2 ds\leq C[A+\Psi(t)]. \tag{2.15}$$

Step 4. If we add (2.11;2) to (2.13;1), then we obtain

$$\frac{d}{dt}\|\sqrt{\rho}\,\nabla^2 v\|_0^2+2\mu\|\nabla^3 v\|_0^2+m\|\nabla v_t\|_0^2+2\mu\frac{d}{dt}\|\nabla^2 v\|_0^2 \tag{2.16}$$

$$\leq C[(1+\|\nabla\rho\|_2^2+\|v\|_3^2)^2+(1+\|\nabla\rho\|_2^2)(\|v_t\|_0^2+\|f\|_3^2)].$$

Hence, due to (2.3) and (2.15),

$$\|\nabla^2 v\|_0^2+\int_0^t\|\nabla v_t\|_0^2 ds+\mu\int_0^t\|\nabla^3 v\|_0^2\, ds\leq C[A+\Psi(t)]^2. \tag{2.17}$$

Step 5. Add (2.11;3) to (2.13;2). Similarly to the above, we have

$$\frac{d}{dt}\|\sqrt{\rho}\,\nabla^3 v\|_0^2+2\mu\|\nabla^4 v\|_0^2+m\|\nabla^2 v_t\|_0^2+2\mu\frac{d}{dt}\|\nabla^3 v\|_0^2 \tag{2.18}$$

$$\leq C[(1+\|\nabla\rho\|_2^2+\|v\|_3^2)^2+(1+\|\nabla\rho\|_2^2)(\|v_t\|_1^2+\|f\|_3^2)],$$

and thus,

$$\|\nabla^3 v\|_0^2+\int_0^t\|\nabla^2 v_t\|_0^2 ds+\mu\int_0^t\|\nabla^4 v\|_0^2 ds\leq C[A+\Psi(t)]^3. \tag{2.19}$$

Consequently, from (2.11;0), (2.15), (2.17), (2.19), we find that (2.9) is deduced. □

Lemma 2.3. *There exists $T_0 \in (0, T]$ independent of μ such that*

(2.20)
$$\Psi(t) \leq A \text{ for } t \leq T_0.$$

Proof. If we set $Y(t) = \Psi(t) + A$, then from Lemma 2.1, 2.2, we have a differential inequality

(2.21)
$$\frac{d}{dt} Y(t) \leq C Y^3(t).$$

Therefore we find that

(2.22)
$$Y(t) \leq \frac{A}{\sqrt{1-2CA^2 t}} \text{ provided } t < (2CA^2)^{-1},$$

and thus

(2.23)
$$\Psi(t) \leq A \text{ for } t \leq T_0 = \frac{3}{8CA^2}.$$
□

3. Proof of Theorem

First, we note that from Lemma 2.1, 2.2, 2.3, the estimate

(3.1)
$$\sup_{0 \leq t \leq T_0} [\|\nabla \rho(t)\|_2^2 + \|v(t)\|_3^2] + \int_0^{T_0} \|v_t(t)\|_2^2 dt + \mu \int_0^{T_0} \|\nabla v(t)\|_3^2 dt \leq C$$

is valid.

The unique solvability of problem (1.1), (1.2) is proved with semi Galerkin method based on (3.1). As this process is parallel with that of [1, Chapter 3], we omit it here and restrict ourselves to establish (1.9).

Subtracting (1.1) with $\mu > 0$ from (1.1) with $\mu = 0$, we get the following linear system of equations for $\sigma = \rho^0 - \rho^\mu, w = v^0 - v^\mu$ and $q = p^0 - p^\mu$:

$$
(3.2) \quad \begin{cases} \sigma_t + v^\mu \cdot \nabla \sigma = -w \cdot \nabla \rho^0, \\ \rho^\mu [w_t + (v^\mu \cdot \nabla) w] + \nabla q = -\rho^\mu (w \cdot \nabla) v^0 + (\nabla p^0/\rho^0)\sigma - \mu \triangle v^\mu, \\ \operatorname{div} w = 0, \\ \sigma|_{t=0} = 0, \\ w|_{t=0} = 0. \end{cases}
$$

In the same way for getting *a priori* estimates, we have from (3.1),

$$
(3.3) \quad \|\sigma(t)\|_2^2 \le C \int_0^t \|w(s)\|_2^2 ds
$$

and

$$
(3.4) \quad \|w(t)\|_2^2 \le C \int_0^t (\|\sigma(s)\|_2^2 + \|w(s)\|_2^2) ds + \mu C T_0.
$$

Hence, by Gronwall's inequality, we find that

$$
(3.5) \quad \|\sigma(t)\|_2^2 + \|w(t)\|_2^2 \le \mu C T_0 \exp(C T_0).
$$

This completes the proof of Theorem.

References

[1] S. N. Antontsev, A. V. Kazhikhov and V. N. Monakhov, *Boundary value problems in mechanics of nonhomogeneous fluids,* North-Holland, Amsterdam•New York•Oxford•Tokyo, 1990.

[2] S. Itoh, *On the vanishing viscosity in the Cauchy problem for the equations of a non-homogeneous incompressible fluid,* Glasgow Math. J. **36** (1994), 123-129.

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