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ON THE EULER EQUATIONS OF NONHOMOGENEOUS INCOMPRESSIBLE PERFECT FLUIDS

ノンホモジニアウスな非圧縮性完全流体のオイラー方程式について

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ABSTRACT. We consider the unique solvability of the initial-boundary value problem to the Euler equations for a nonhomogeneous incompressible fluid in a bounded or unbounded domain in $\mathbb{R}^3$.

§1. Introduction

Let Ω be a bounded or unbounded domain in $\mathbb{R}^3$ with a smooth boundary S. We consider the system of equations

$$(1.1)\qquad \begin{cases} \rho_t + v\cdot\nabla\rho = 0, \\ \rho[v_t + (v\cdot\nabla)v] + \nabla p = \rho f, \\ \operatorname{div} v = 0, \end{cases}$$

in $Q_T = \Omega\times[0,T]$, $T>0$, where $f(x,t)$ is a given vector field of external forces, while the density $\rho(x,t)$, the velocity vector $v(x,t)$ and the pressure $p(x,t)$ are the unknowns.

In this paper, we solve (1.1) under the following initial-boundary conditions:

$$(1.2)\qquad \begin{cases} v\cdot n|_{S_T} = 0, \\ \rho|_{t=0} = \rho_0(x), \\ v|_{t=0} = v_0(x), \end{cases}$$

where n is the unit outward normal to S, and $S_T = S\times[0,T]$.

In the previous paper [2], [3], we discuss the problem in the case $\Omega = \mathbb{R}^3$.

Our theorem is the following.

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Theorem. *Assume that*

$$\rho_0(x)\in C^0(\bar{\Omega}),\quad \nabla\rho_0(x)\in H^2(\Omega),\quad 0<m\leq\rho_0(x)\leq M<\infty, \tag{1.3}$$

$$v_0(x)\in H^3(\Omega),\quad v_0\cdot n|_S=0,\quad \mathrm{div}\ v_0=0, \tag{1.4}$$

$$f(x,t)\in C^0([0,T];\ H^3(\Omega)). \tag{1.5}$$

Then there exists $T_0\in(0,T]$ *such that problem* (1.1), (1.2) *has a unique solution* $(\rho,v,p)(x,t)$ *which satisfies*

$$\rho(x,t)\in C^0(\bar{Q}_{T_0}),\ \nabla\rho(x,t)\in C^0([0,T_0];\ H^2(\Omega)),\ 0<m\leq\rho(x,t)\leq M<\infty, \tag{1.6}$$

$$v(x,t)\in C^0([0,T_0];\ H^3(\Omega)), \tag{1.7}$$

$$\nabla p(x,t)\in C^0([0,T_0];\ H^3(\Omega)). \tag{1.8}$$

§2. Auxiliary Problems

We assume that $v(x,t)\in C^0([0,T];\ H^3(\Omega))$ is a given vector field such that div $v=0$ and $v\cdot n|_{S_T}=0$. Hereafter c_j's are the positive constants depending only on the imbedding theorems.

Lemma 2.1. *Under the assumption* (1.3), *problem*

$$\begin{cases}\rho_t+v\cdot\nabla\rho=0,\\ \rho|_{t=0}=\rho_0(x),\end{cases} \tag{2.1}$$

has a unique solution $\rho(x,t)\in C^0(\bar{Q}_T)$ *with* $\nabla\rho(x,t)\in C^0([0,T];\ H^2(\Omega))$, *which satisfies the estimates*

$$m\leq\rho(x,t)\leq M \tag{2.2}$$

and

$$\frac{d}{dt}\|\nabla\rho(t)\|_2 \le c_1\|\nabla v(t)\|_2\|\nabla\rho(t)\|_2, \tag{2.3}$$

where $\|\cdot\|_k = \|\cdot\|_{H^k(\Omega)}$.

Moreover, if we put $\xi(x,t)=\rho(x,t)^{-1}$, *then the estimates*

$$M^{-1}\le\xi(x,t)\le m^{-1} \tag{2.4}$$

and

$$\frac{d}{dt}\|\nabla\xi(t)\|_2 \le c_1\|\nabla v(t)\|_2\|\nabla\xi(t)\|_2 \tag{2.5}$$

are valid.

Proof. The way to derive (2.2) and (2.3) is just like that of Lemma 2.1 in [3]. If we note that $\xi(x,t)$ satisfies the equation

$$\begin{cases} \xi_t + v\cdot\nabla\xi = 0, \\ \xi|_{t=0} = \rho_0(x)^{-1} \equiv \xi_0(x), \end{cases}$$

the estimates (2.4) and (2.5) directly follow from (2.2) and (2.3). □

Lemma 2.2. *Let* $\rho(x,t)$ *be the unique solution of* (2.1) *guaranteed in Lemma 2.1 and* $f(x,t)\in C^0([0,T];\ H^3(\Omega))$. *Then problem*

$$\begin{cases} \operatorname{div}(\xi\nabla p) = \operatorname{div} f - \sum_{i,j=1}^{3} v^i_{x_j} v^j_{x_i} \equiv F, \\ \xi\frac{\partial p}{\partial n}|_S = f\cdot n + \sum_{i,j=1}^{3} v^i v^j \phi^{ij} \equiv G, \quad \phi^{ij} = n^j_{x_i}, \end{cases} \tag{2.6}$$

has a unique solution $p(x,t)$ *with* $\nabla p(x,t)\in C^0([0,T];\ H^3(\Omega))$, *satisfying*

$$\|\nabla p(t)\|_3 \le K_1(\|\nabla\xi(t)\|_2)\left(\|f(t)\|_3 + \|v(t)\|_3^2\right), \tag{2.7}$$

where K_1 *is a nondecreasing function of* $\|\nabla\xi(t)\|_2$, *depending on m and M. Hereafter,* K_j*'s are functions, having the same properties as* K_1.

Proof. We first note that $(2.6)_1$ comes from applying the divergence operator on both sides of $(1.1)_2$ and $(2.6)_2$ from taking the scalar product of each side of $(1.1)_2$ with n (cf. Temam[5]). It is well-known from Agmon-Douglis-irenberg [1] that problem (2.6) is solvable in $H^3(\Omega)$ and the estimate

$$\|\nabla p(t)\|_3 \leq K_2\left(\|\xi\|_{C^1(\bar{\Omega})}\right)\left(\|F\|_3+\|G\|_{H^{5/2}(S)}\right)$$

is valid. Hence we can immediately get (2.7). □

Lemma 2.3. *Let* $\rho(x,t)$ *and* $f(x,t)$ *be the same as in Lemma 2.2 and* $p(x,t)$ *the unique solution of* (2.6). *Then problem*

$$(2.8)\qquad \begin{cases} u_t+(v\cdot\nabla)u=-\xi\nabla p+f, \\ u|_{t=0}=v_0(x), \end{cases}$$

has a unique solution $u(x,t)\in C^0([0,T];\, H^3(\Omega))$. *Moreover,* $u(x,t)$ *satisfies*

$$(2.9)\qquad \frac{d}{dt}\|u(t)\|_3 \leq c_2\|v(t)\|_3\|u(t)\|_3+K_3(\|\nabla\xi(t)\|_2)\left(\|f(t)\|_3+\|v(t)\|_3^2\right).$$

Proof. Referring to Lemma 2.1, we should only estimate the term

$$\sum_{|\alpha|=0}^{3}\int_\Omega D_x^\alpha(f-\xi\nabla p)\cdot D_x^\alpha u\,dx.$$

Since

$$\begin{aligned}\sum_{|\alpha|=0}^{3}\int_\Omega |D_x^\alpha(\xi\nabla p)|\,|D_x^\alpha u|\,dx &\leq m^{-1}\|\nabla p\|_3\|u\|_3+\|\nabla\xi\|_2\|\nabla p\|_2\|\nabla u\|_2 \\ &\leq K_3(\|\nabla\xi\|_2)\left(\|f\|_3+\|v\|_3^2\right)\|u\|_3,\end{aligned}$$

the desired estimate is obtained. □

§3. Successive Approximations

In order to prove Theorem, we use the method of successive approximations in the following form:

$$v^{(0)}=0, \tag{3.1}$$

and for $k=1,2,3,\cdots$, $\rho^{(k)}$, $p^{(k)}$ and $u^{(k)}$ are, respectively, the solutions of problems

$$\begin{cases} \rho_t^{(k)}+v^{(k-1)}\cdot\nabla\rho^{(k)}=0, \\ \rho^{(k)}|_{t=0}=\rho_0(x), \end{cases} \tag{3.2}$$

$$\begin{cases} \operatorname{div}(\xi^{(k)}\nabla p^{(k)})=\operatorname{div} f-\sum_{i,j=1}^{3} v_{x_j}^{(k-1),i}v_{x_i}^{(k-1),j}, \\ \xi^{(k)}\dfrac{\partial p^{(k)}}{\partial n}\Big|_S=f\cdot n+\sum_{i,j=1}^{3} v^{(k-1),i}v^{(k-1),j}\phi^{ij}, \quad \xi^{(k)}=\left(\rho^{(k)}\right)^{-1}, \end{cases} \tag{3.3}$$

and

$$\begin{cases} u_t^{(k)}+(v^{(k-1)}\cdot\nabla)u^{(k)}=-\xi^{(k)}\nabla p^{(k)}+f, \\ u^{(k)}|_{t=0}=v_0(x). \end{cases} \tag{3.4}$$

Finally, let

$$v^{(k)}=u^{(k)}-\nabla\psi^{(k)}, \tag{3.5}$$

where $\psi^{(k)}$ is the solution of problem

$$\begin{cases} \triangle\psi^{(k)}=\operatorname{div} u^{(k)}, \\ \dfrac{\partial\psi^{(k)}}{\partial n}\Big|_S=u^{(k)}\cdot n. \end{cases} \tag{3.6}$$

Lemma 3.1. *The sequence* $\{v^{(k)}\}_k$ *is bounded in* $C^0([0,T_0];H^3(\Omega))$ *for a sufficiently small* $T_0\in(0,T]$.

Proof. From the consequences in section 2, we obtain

$$\|v^{(k)}(t)\|_3\le c_3\exp\left(c_2\int_0^t\|v^{(k-1)}(s)\|_3ds\right) \tag{3.7}$$

$$\times\Big[\|v_0\|_3+\int_0^t K_3\left(\left\|\nabla\xi^{(k)}(s)\right\|_2\right)\left(\|f(s)\|_3+\left\|v^{(k-1)}(s)\right\|_3^2\right)ds\Big],$$

since

$$\left\|v^{(k)}(t)\right\|_3\leq\left\|u^{(k)}(t)\right\|_3+\left\|\nabla\psi^{(k)}(t)\right\|_3\leq c_4\left\|u^{(k)}(t)\right\|_3.$$

Let us choose

$$A\geq 2c_3\Big[\|v_0\|_3+K_3(2\|\nabla\xi_0\|_2)\left(T\|f\|_{C^0([0,T];\,H^3(\Omega))}+1\right)\Big],$$

and define

$$T_0=\min\left\{(c_1A)^{-1}\log2,\ \ (c_2A)^{-1}\log2,\ \ A^{-2}\right\}.$$

Then we find that

$$\sup_{0\leq t\leq T_0}\left\|v^{(k)}(t)\right\|_3\leq A\ \ \text{provided that}\ \ \sup_{0\leq t\leq T_0}\left\|v^{(k-1)}(t)\right\|_3\leq A.$$

Therefore, by induction, we have the assertion of the lemma. □

By the direct calculation, we get

Lemma 3.2. *For* $k=1,2,3,\cdots$, the estimates

$$\sup_{0\leq t\leq T_0}\left\|\nabla\rho^{(k)}(t)\right\|_2\leq 2\|\nabla\rho_0\|_2,\quad \sup_{0\leq t\leq T_0}\left\|\rho_t^{(k)}(t)\right\|_2\leq 2\|\nabla\rho_0\|_2A,$$

$$\sup_{0\leq t\leq T_0}\left\|\nabla\xi^{(k)}(t)\right\|_2\leq 2\|\nabla\xi_0\|_2,\quad \sup_{0\leq t\leq T_0}\left\|\xi_t^{(k)}(t)\right\|_2\leq 2\|\nabla\xi_0\|_2A,$$

$$\sup_{0\leq t\leq T_0}\left\|\nabla p^{(k)}(t)\right\|_3\leq K_1(2\|\nabla\xi_0\|_2)\left(\|f\|_{C^0([0,T];H^3(\Omega))}+A^2\right)\equiv B$$

and

$$\sup_{0\leq t\leq T_0}\left\|u_t^{(k)}(t)\right\|_2\leq A^2+m^{-1}B(m^{-1}+2\|\nabla\xi_0\|_2)+\|f\|_{C^0([0,T];H^3(\Omega))}$$

hold.

§4. Proof of Theorem

Set $\sigma^{(k)}=\rho^{(k)}-\rho^{(k-1)}$, $\eta^{(k)}=\xi^{(k)}-\xi^{(k-1)}$, $h^{(k)}=u^{(k)}-u^{(k-1)}$, $q^{(k)}=p^{(k)}-p^{(k-1)}$ and $w^{(k)}=v^{(k)}-v^{(k-1)}$. Then we have

$$\begin{cases} \sigma_t^{(k)}+v^{(k-1)}\cdot\nabla\sigma^{(k)}=-w^{(k-1)}\cdot\nabla\rho^{(k-1)}, \\ \sigma^{(k)}|_{t=0}=0, \end{cases} \tag{4.1}$$

$$\begin{cases} \eta_t^{(k)}+v^{(k-1)}\cdot\nabla\eta^{(k)}=-w^{(k-1)}\cdot\nabla\xi^{(k-1)}, \\ \eta^{(k)}|_{t=0}=0, \end{cases} \tag{4.2}$$

(4.3)

$$\begin{cases} \operatorname{div}(\xi^{(k)}\nabla q^{(k)})=-\operatorname{div}(\eta^{(k)}\nabla p^{(k-1)})-\sum_{i,j=1}^{3}\left(w_{x_j}^{(k-1),i}v_{x_i}^{(k-1),j}+v_{x_j}^{(k-2),i}w_{x_i}^{(k-1),j}\right), \\ \xi^{(k)}\dfrac{\partial q^{(k)}}{\partial n}\Big|_S=\sum_{i,j=1}^{3}\left(w^{(k-1),i}v^{(k-1),j}+v^{(k-2),i}w^{(k-1),j}\right)\phi^{ij}-\eta^{(k)}\dfrac{\partial p^{(k-1)}}{\partial n}\Big|_S, \end{cases}$$

and

$$\begin{cases} h_t^{(k)}+(v^{(k-1)}\cdot\nabla)h^{(k)}+\xi^{(k)}\nabla q^{(k)}=-(w^{(k-1)}\cdot\nabla)u^{(k-1)}-\eta^{(k)}\nabla p^{(k-1)}, \\ h^{(k)}|_{t=0}=0. \end{cases} \tag{4.4}$$

Let L be the generic constant depending on m, M, $\|\nabla\rho_0\|_2$, $\|v_0\|_3$, $\|f\|_{C^0([0,T];H^3(\Omega))}$ and T, then, in the same way used for getting the estimates of ρ, p and u, we get

$$\left\|\sigma^{(k)}(t)\right\|_2\leq L\int_0^t\left\|w^{(k-1)}(s)\right\|_2ds,$$

$$\left\|\eta^{(k)}(t)\right\|_2\leq L\int_0^t\left\|w^{(k-1)}(s)\right\|_2ds,$$

$$\left\|\nabla q^{(k)}(t)\right\|_2\leq L\left(\left\|\eta^{(k)}(t)\right\|_2+\left\|w^{(k-1)}(t)\right\|_2\right)$$

and

$$\left\|h^{(k)}(t)\right\|_2 \leq L\int_0^t \left(\left\|\eta^{(k)}(s)\right\|_2 + \left\|\nabla q^{(k)}(s)\right\|_2 + \left\|w^{(k-1)}(s)\right\|_2\right) ds.$$

From these inequalities, since

$$\left\|w^{(k)}(t)\right\|_2 \leq c_5 \left\|h^{(k)}(t)\right\|_2,$$

it follows that

$$\left\|w^{(k)}(t)\right\|_2 \leq L\int_0^t \left\|w^{(k-1)}(s)\right\|_2 ds \leq L^{k-1}\frac{t^{k-1}}{(k-1)!}\sup_{0\leq s\leq t}\left\|w^{(1)}(s)\right\|_2.$$

Consequently, we have

$$\sup_{0\leq t\leq T_0}\left\|w^{(k)}(t)\right\|_2 \leq AL^{k-1}\frac{T_0^{k-1}}{(k-1)!}$$

Therefore we find that

$$\sum_{k=1}^{\infty}\left\|w^{(k)}\right\|_{C^0([0,T_0];H^2(\Omega))} < \infty.$$

This implies that $(\rho^{(k)}, p^{(k)}, u^{(k)}, v^{(k)})(x,t) \to (\rho,p,u,v)(x,t)$ as $k\to\infty$, which satisfies the equations

$$(4.5)\quad \begin{cases} \rho_t + v\cdot\nabla\rho = 0, \\ \operatorname{div}((v\cdot\nabla)v + \rho^{-1}\nabla p - f) = 0, \\ u_t + (v\cdot\nabla)u + \rho^{-1}\nabla p = f, \\ \triangle\psi = \operatorname{div} u, \\ v = u - \nabla\psi, \\ ((v\cdot\nabla)v + \rho^{-1}\nabla p - f)\cdot n|_{S_{T_0}} = 0, \\ (u-\nabla\psi)\cdot n|_{S_{T_0}} = 0, \\ \rho|_{t=0} = \rho_0(x), \\ u|_{t=0} = v_0(x). \end{cases}$$

Now let us show that $u = v$. Applying the divergence operator on both sides of $(4.5)_3$ and taking into account $(4.5)_{2,4,5}$, we get

$$(4.6)\qquad (\operatorname{div} u)_t + v \cdot \nabla (\operatorname{div} u) = -\sum_{i,j=1}^{3} v^i_{x_j} \psi_{x_i x_j}.$$

If we take the scalar product of each side of $(4.5)_3$ with n, we obtain

$$(4.7)\qquad (u \cdot n)_t + v \cdot \nabla (u \cdot n) = \sum_{i,j=1}^{3} v^i \psi_{x_j} \phi^{ij}.$$

Nothing that div $v=0$, $v \cdot n|_{S_{T_0}}=0$ and

$$\|\psi\|_3 \le c_6 \left(\|\operatorname{div} u\|_0 + \|u \cdot n\|_{H^{1/2}(S)} \right),$$

we have the inequality

$$(4.8)\qquad \|\operatorname{div} u\|_0 + \|u \cdot n\|_{H^{1/2}(S)} \le L \int_0^t \left(\|\operatorname{div} u\|_0 + \|u \cdot n\|_{H^{1/2}(S)} \right) ds,$$

which means div $u=0$ and $u \cdot n|_{S_{T_0}}=0$.

This completes the proof of Theorem.

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