Holomorphic Distributions on a Generic Submanifold

ジェネリック部分多様体上の正則分布

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Abstract

An integral formula for the Ricci tensor of a generic submanifold immersed in a complex projective space is given. As an application, the relation of the integrability condition of the holomorphic distribution and the Ricci curvature of a compact generic submanifold immersed in a complex projective space is studied.

Key words: Ricci tensor, holomorphic distribution, generic submanifold

Introduction.

The purpose of the present paper is to study the relation of the holomorphic distribution and the Ricci curvature on a generic submanifold of a complex projective space.

Let M be an n-dimensional generic submanifold of a a complex m-dimensional projective space CP^m with almost complex structure J and Hermitian metric g. Then $JT_x(M)^{\perp} \subset T_x(M)$, where $T_x(M)$ and $T_x(M)^{\perp}$ denote the tangent space and the normal space of M, redpectively. Any real hypersurface is obviousely a generic submanifold. The holomorphic distribution H on M is defined to be $H_x = \{X | X \in T_x(M), g(X, JV)=0, V \in T_x(M)^{\perp}\}$ for $x \in M$.

In [1], Bejancu-Deshmukh proved that if the Ricci tensor S of a compact real hypersurface M of CP^m satisfies $S(\xi, \xi) \geq 0$, then H is not integrable, where ξ denotes the structure vector of M. We improved this result to the case that the ambient manifold is a Nearly Kaehler manifold (see [2]).

In this paper, we study the integrability condition of H on a generic submanifold and give an integral formula for the Ricci tensor and the second fundamental form of a generic submanifold M of CP^m. Our result give a generalization of the one in Bejancu-Deshmukh [1] of a real hypersurface of a complex projective space.

1. Generic submanifolds.

Let CP^m denote the complex projective space of complex dimension m (real dimension $2m$) with constant holomorphic sectional curvature four. We denote by J the almost complex structure

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of CP^m. The Hermitian metric of CP^m is denoted by G.

Let M be a real n-dimensional Riemannian manifold isometrically immersed in CP^m. We denote by g the Riemannian metric induced on M from G.

We denote by $T_x(M)$ and $T_x(M)^{\perp}$ the tangent space and the normal space of M respectively. If $JT_x(M)^{\perp} \subset T_x(M)$ for any point x of M, then we call M a *generic submanifold* of CP^m. Any real hypersurface of CP^m is obviousely a generic submanifold of CP^m.

We denote by $\tilde{\nabla}$ the operator of covariant differentiation in CP^n, and by ∇ the one in M determined by the induced metric. Then the *Gauss and Weingarten formulas* are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X, Y), \qquad \tilde{\nabla}_X V = -A_V X + D_X V$$

for any vector fields X and Y tangent to M and any vector field V normal to M, where D denotes the operator of covarinat differentiation with respect to the linear connection induced in the normal bundle $T(M)^{\perp}$ of M. We call both A and B the *second fundamental form* of M and are related by $G(B(X, Y), V) = g(A_V X, Y)$. The second fundamental form A and B are symmetric. A_V can be considered as a (n, n)-matrix.

The covariant derivative $(\nabla_X A)_V Y$ of A is defined to be

$$(\nabla_X A)_V Y = \nabla_X (A_V Y) - A_{D_X V} Y - A_V \nabla_X Y.$$

If $(\nabla_X A)_V Y = 0$ for any vector fields X and Y tangent to M, then the second fundamental form A of M is said to be *parallel in the direction of the normal vector V*. If the second fundamental form is parallel in any direction, it is said to be *parallel*. If $\mathrm{Tr} A_V = 0$ for any vector V normal to M, then M is said to be *minimal*, where Tr denotes the trace of the operator. A vector field V normal to M is said to be *parallel* if $D_X V = 0$ for any vector field X tangent to M.

In the sequel, we assume that M is a generic submanifold of CP^m. The tangent space $T_x(M)$ of M is decomposed as $T_x(M) = H_x(M) + JT_x(M)^{\perp}$ at each point x of M, where $H_x(M)$ denotes the orthogonal complement of $JT_x(M)^{\perp}$ in $T_x(M)$. Then we see that $H_x(M)$ is a holomorphic subspace of $T_x(M)$.

For any vector field X tangent to M, we put

$$JX = PX + FX,$$

where PX is the tangential part of JX and FX the normal part of JX. Then P is an endomorphism on the tangent bundle $T(M)$ and F is a normal bundle valued 1-form on the tangent bundle $T(M)$.

We put $JV = tV$ for a vector field V normal to M. Then w have $P^2 = -I - tF$ and $FP = 0$. We define the covariant derivatives of P and F by $(\nabla_X P) Y = \nabla_X (PY) - P\nabla_X Y$ and $(\nabla_X F) Y = D_X (FY) - F\nabla_X Y$, respectively. We then have

$$(\nabla_X P) Y = A_{FY} X + tB(X, Y), \qquad (\nabla_X F) Y = -B(X, PY),$$

$$\nabla_X tV = -PA_V X + tD_X V, \qquad B(X, tV) = -FA_V X.$$

For any vector fields X and Y in $JT(M)^{\perp}$ we obtain

$$A_{FX}Y = A_{FY}X.$$

We also have

$$A_V tU = A_U tV$$

for any vector fields U and V normal to M.

We denote by R the Riemannian curvature tensor field of M. Then the *equation of Gauss* is given by

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(PY, Z)PX - g(PX, Z)PY$$
$$- 2g(PX, Y)PZ + A_{B(Y, Z)}X - A_{B(X, Z)}Y,$$

for any X, Y and Z tangent to M.

The *equation of Codazzi* of M is given by

$$g((\nabla_X A)_V Y, Z) - g((\nabla_Y A)_V X, Z)$$
$$= g(PX, Z)g(Y, tV) - g(PY, Z)g(X, tV) - 2g(X, PY)g(Z, tV).$$

We denote by S the Ricci tensor field of M. Then

$$S(X, Y) = (n-1)g(X, Y) + 3g(PX, PY)$$
$$+ \sum \mathrm{Tr}A_a\, g(A_a X, Y) - \sum g(A_a^2 X, Y),$$

where A_a is the second fundamental form in the direction of v_a, $\{v_1, \cdots, v_p\}$ being an orthonormal frame for $T_x(M)^{\perp}$. From this the scalar curvature r of M is given by

$$r = (n-1)n + 3(n-p) + \sum(\mathrm{Tr}A_a)^2 - \sum \mathrm{Tr}A_a^2,$$

where p is the codimension of M, that is, $p = 2m - n$.

We define the curvature tensor $R^{\perp}$ of the normal bundle of M by

$$R^{\perp}(X, Y)V = D_X D_Y V - D_Y D_X V - D_{[X, Y]}V.$$

Then we have

$$g(R^{\perp}(X, Y)V, U) + g([A_U, A_V]X, Y)$$
$$= g(Y, tV)g(X, tU) - g(X, tV)g(Y, tU).$$

If $R^{\perp}$ vanishes identically, the normal connection of M is said to be flat. We can see that the normal connection of M is flat if and only if there exist locally p mutually orthogonal unit normal vector fields v_a such that each of the v_a is parallel.

Generally, we have the following formula (see [4])

$$\operatorname{div}(\nabla_X X) - \operatorname{div}((\operatorname{div}X)X)$$
$$= S(X, X) + \frac{1}{2}|L(X)g|^2 - |\nabla X|^2 - (\operatorname{div}X)^2.$$

If U is a parallel section in the normal bundle of a generic submanifold M, then

$$\nabla_X tU = (\nabla_X t)U = -PA_U X.$$

Hence we have

$$\operatorname{div}(tU) = -\operatorname{Tr}PA_U = 0.$$

This implies

$$\operatorname{div}(\nabla_{tU} tU) = S(tU, tU) + \frac{1}{2}|L(tU)g|^2 - |\nabla tU|^2.$$

We notice here that the following equations hold.

$$S(tU, tU) = (n-1)g(tU, tU)$$
$$+ \sum \operatorname{Tr}A_a g(A_a tU, tU) - \sum g(A_a^2 tU, tU),$$
$$|\nabla tU|^2 = \operatorname{Tr}A_U^2 - \sum_a g(A_a^2 tU, tU),$$
$$(L(tU)g)(X, Y) = g(\nabla_X tU, Y) + g(\nabla_Y tU, X)$$
$$= -g((PA_U - A_U P)X, Y).$$

2. Holomorphic distribution.

Let M be a real n-dimensional Riemannian manifold isometrically immersed in CP^m. We consider the holomorphic distribution H defined by

$$H : x \to H_x = T_x(M) \cap JT_x(M).$$

We see that $\dim H_x = n - p$, where p is the codimension of M. In the following, we take an orthonormal basis $\{e_1,...,e_{n-p}, e_{n-p+1} = tv_1, ..., e_n = tv_p\}$, where $\{v_a\}$ is an orthonormal basis of $T_x(\mathrm{M})^\perp$. We use the convention that the ranges of indices are respectively:

$$k, l, s = 1, ..., n-p; \qquad a, b, c = 1, ..., p.$$

Let $X, Y \in H$. For any vector field V normal to M, we have

$$g([X, Y], tV) = g(Y, PA_V) - g(X, PA_V Y).$$

Consequently, H is integrable if and only if $g((PA_V + A_V P)X, Y) = 0$.

We suppose that H is integrable. Let U be a parallel secion in the normal bundle of M. Then

$$\operatorname{div}(-PA_U tU) = S(tU, tU) + \frac{1}{2}|[P, A_U]|^2 - |\nabla tU|^2,$$

$$\frac{1}{2}|[P, A_U]|^2 = \sum_k (g(A_U e_k, A_U e_k) + g(PA_U e_k, PA_U e_k)),$$

where $\{e_k\}$ is an orthonormal basis for H_x.

Thus we have

$$\operatorname{div}(-PA_u tU) = S(tU, tU) + \sum_k g(PA_U e_k, PA_U e_k),$$

where we have used

$$|\nabla tU|^2 = \operatorname{Tr} A_U^2 - \sum_a g(A_a^2 tU, tU) = \sum_k g(A_U e_k, A_U e_k).$$

Proposition 1. Let M be a compact generic submanifold of CP^m with parallel section U in the normal bundle. If the holomorphic distribution H is integrable, then

$$\int_M [S(tU, tU) + \sum g(PA_U e_k, PA_U e_k)] = 0.$$

Moreover, if $S(tU, tU) \geq 0$, then $S(tU, tU) = 0$ and $PA_U e_k = 0$ for all $e_k \in H_x$.

Lemma 1. Let M be a generic submanifold of CP^m. If the holomorphic distribution H is integrable, then

$$\sum_k g(A_U e_k, e_k) = 0.$$

Proof. For any $X, Y \in H_x$, $g(PA_U X, Y) + g(A_U PX, Y) = 0$. Hence we have

$$\sum_k g(A_U e_k, e_k) + \sum_k g(A_U Pe_k, Pe_k) = 2\sum_k g(A_U e_k, e_k) = 0.$$

From Lemma 1 we have

$$S(tU, tU) = (n-1)\, g(tU, tU) + \sum_{a,b} g(A_a tv_b, tv_b)\, g(A_a tU, tU)$$

$$- \sum g(A_a^2 tU, tU).$$

Lemma 2. Let M be a generic submanifold of CP^m and U, V be unit normal vector fields. If $DV = 0$, then

$$g([A_V, A_U]X, Y) = g(Y, tU)\, g(X, tV) - g(X, tU)\, g(Y, tV),$$

$$\sum_{a,i} g([A_V, A_a]e_i, [A_V, A_a]e_i) = 2(p-1).$$

Proof. Since $R^{\perp} = 0$, we see

$$g([A_V, A_U]X, Y) = g(Y, tU)\, g(X, tV) - g(X, tU)\, g(Y, tV),$$

from which

$$[A_V, A_U]X = g(X, tV)\, tU - g(X, tU)\, tV.$$

From this we have the second equation.

Lemma 3. Let M be a generic submanifold of CP^m. If U is a parallel unit normal vector field, then

$$\sum_{a,} [g(A_U tv_a, A_U] tv_a) - g(A_V tU, A_a tv_a)] = p-1.$$

Proof. In the proof of Lemma 2, we see

$$T(V, U, X) = [A_V, A_U]X - g(X, tV)tU + g(X, tU)tV = 0.$$

Hence, we get

$$\begin{aligned} 0 &= \sum_{a,i} g(T(v_a, U, e_i), T(v_a, U, e_i)) \\ &= \sum_{a,i} g([A_U, A_a]e_i, [A_U, A_a]e_i) + 2(p-1) \\ &+ 4\sum_a (g(A_a tv_a, A_U U) - g(A_U tv_a, A_U tv_a)). \end{aligned}$$

From this and Lemma 2 we have our equation.

Lemma 4. Let M be a generic submanifold of CP^m with flat normal connection. If the holomorphic distribution H is integrable and $S(tU, tU) \geq 0$ for a parallel unit normal vector field U, then

$$\sum_{b,k} g(A_b tv_b, e_k) g(e_k, A_U tU) = n - p.$$

Proof. Since $S(tU, tU)$=0, we have, by Lemmas 1 and 3,

$$\begin{aligned} (n-1) &= \sum_a g(A_a tU, A_a tU) - \sum_{a,b} g(A_a tv_b, tv_b) g(A_a tU, tU) \\ &= \sum_a g(A_U tU, A_a tv_a) + (p-1) - \sum_{a,b} g(A_a tv_b, tv_b) g(A_a tU, tU). \end{aligned}$$

Hence we have

$$\sum_a g(A_U tU, A_a tv_a) - \sum_{a,b} g(A_a tv_b, tv_b) g(A_a tU, tU) = n - p.$$

On the other hand, we see

$$\begin{aligned} \sum_{a,b} g(A_a tv_b, tv_b) g(A_a tU, tU) &= \sum_{a,b} g(A_b tv_b, tv_a) g(A_U tU, tv_a) \\ &= \sum_b g(A_b tv_b, A_U tU) - \sum_{b,k} g(A_b tv_b, e_k) g(e_k, A_U tU). \end{aligned}$$

Thus we have our assertion.

Lemma 5. Let M be a generic submanifold of CP^m with flat normal connection. Then

$$g(A_a tv_b, A_a tv_b) = g(A_a tv_a, A_b tv_b) + 1, \qquad a \neq b.$$

Proof. From Lemma 2, we see

$$g([A_V, A_U] tv_b, tv_c) = g(tv_c, tU) g(tv_a, tV) - g(tv_a, tU) g(tv_c, tV),$$

which proves our equation.

Lemma 6. Let M be a generic submanifold of CP^m with flat normal connection. If the holomorphic distribution H is integrable and $S(tU, tU) \geq 0$, then

$$\sum_{b,k} g(A_b tv_b, e_k)\, g(e_k, A_U tV) = 0, \qquad \textit{where} \quad g(U, V) = 0,$$

U, V being unit normal vectors.

Proof. We use Lemma 4. Since $DU = DV = 0$, we see that $U + V$ is also parallel in the normal bundle. Then

$$\sum_{a,b} g(A_b tv_b, e_k)\, g(e_k, A_{(U+V)/\sqrt{2}}(tU + tV)/\sqrt{2}) = n - p.$$

From

$$A_{(U+V)/\sqrt{2}}(tU + tV)/\sqrt{2} = \frac{1}{2}(A_U tU + 2A_U tV + A_V tV),$$

we have our equation.

Remark. Lemmas 2,3 and 5 follow from the quite similar method of Lemma 2.3 in [3].

3. Theorems.

First of all, we give an equation which improve the result of [1].

Theorem 1. Let M be a compact generic submanifold of CP^m with flat normal connection. If the holomorphic distribution H is integrable and if $S(tU, tU) \geq 0$ for any unit vector field U normal to M, then

$$\sum_{a,b} g(A_b^2 e_s, PA_a tv_a) - n \sum_a g(PA_a tv_a, e_s) = 0$$

for any $e_s \in H_x$.

Proof. We take the covariant differentiation of

$$\sum_{a,b,k} g(A_b tv_b, e_k)\, g(e_k, A_a tv_a) = (n - p)p,$$

by $e_s \in H_x$, we have

$$\begin{aligned}
0 = & \sum_{a,b,k} g((\nabla_{e_s} A_b) tv_b, e_k)\, g(e_k, A_a tv_a) + \sum_{a,b,k} g(A_b(-PA_b e_s), e_k)\, g(e_k, A_a tv_a) \\
& + \sum_{a,b,k} g(A_a tv_b, \nabla_{e_s} e_k)\, g(e_k, A_a tv_a) \\
= & \sum_{a,b,k} g((\nabla_{tv_b} A_b) e_s, e_k)\, g(e_k, A_a tv_a) + \sum_{a,b,k} g(tv_b, tv_b)\, g(Pe_s, e_k)\, g(e_k, A_a tv_a) \\
& + \sum_{a,b,k,i} g(A_b tv_b, e_l)\, g(e_l, \nabla_{e_s} e_k)\, g(e_k, A_a tv_a)
\end{aligned}$$

$$= \sum_{a,b,k} g((\nabla_{tv_b} A_b) e_s, e_k) g(e_k, A_a tv_a) - p \sum_a g(PA_a tv_a, e_s)$$

On the other hand, we see, using Lemma 6,

$$\sum_{a,b,k} g((\nabla_{tv_b} A_b) e_s, e_k) g(e_k, A_a tv_a)$$

$$\sum_{a,b,k} g(\nabla_{tv_b} A_b e_s, e_k) g(e_k, A_a tv_a) - \sum_{a,b,k} g(A_b \nabla_{tv_b} e_s, e_k) g(e_k, A_a tv_a)$$

$$= - \sum_{a,b,k} g(A_b e_s, \nabla_{tv_b} e_k) g(e_k, A_a tv_a) - \sum_{a,b,k} g(A_b \nabla_{tv_b} e_s, e_k) g(e_k, A_a tv_a)$$

$$= - \sum_{a,b,c} g(A_b e_s, tv_c) g(tv_c, \nabla_{tv_b} e_k) g(e_k, A_a tv_a)$$

$$- \sum_{a,b,c,k} g(A_b \nabla_{tv_c}, e_k) g(tv_c, \nabla_{tv_b} e_s) g(e_k, A_a tv_a)$$

$$= \sum_{a,b,c,k} g(A_b e_s, tv_c) g(- PA_c tv_b, e_k) g(e_k, A_a tv_a)$$

$$+ \sum_{a,b,c,k} g(A_b tv_c, e_k) g(- PA_c tv_b, e_s) g(e_k, A_a tv_a)$$

$$= \sum_{a,b,c,k} g(A_b e_s, tv_c) g(- PA_b tv_c, e_k) g(e_k, A_a tv_a)$$

$$+ \sum_{a,b=c,k} g(A_b tv_c, e_k) g(- PA_c tv_b, e_s) g(e_k, A_a tv_a)$$

$$= \sum_{a,b} g(A_b^2 e_s, PA_a tv_a) - (n - p) \sum_b g(PA_b tv_b, e_s)$$

From these equations we have our result.

In the following, we put $h = \sum_a PA_a tv_a$. The vector field h is in H_x. Since $PA_a e_k = 0$ for all k, if M is a real hypersurface, then $Ah = 0$.

Theorem 2. Let M be an n-dimensional compact generic submanifold of CP^m $(n \neq p)$ with flat normal connection. If the Ricci tensor S of M satisfies $S(tU, tU) \geq 0$ for any vector field U normal to M and $A_a h = 0$ for all a, then the holomorphic distribution H is not integrable.

Proof. From the assumption $n \neq p$, by Lemma 4, we see that h is not vanish. Therefore, if H is integrable, Theorem 1 implies a contradiction.

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