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The scalar curvature of minimal submanifolds in a complex projective space

複素射影空間の極小部分多様体のスカラー曲率

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Abstract

We give a characterization for certain compact minimal CR submanifolds of complex projective space with respect to the condition for the scalar curvature and the dimension of the holomorphic tangent space on the submanifold when the normal connection is flat.

Key words: CR submanifold, scalar curvature, complex projective space, minimal submanifold

1. Introduction.

Minimal submanifolds in complex space forms have been an active field of research for years. The purpose of the present paper is to study a pinching problem in terms of scalar curvatures of minimal submanifolds in a complex projective space.

In 1968, Simons [5] gave the integral formula for the square of the length of the second fundamental form A of a compact n-dimensional minimal submanifold M in a real space form $M^m(k)$ of constant curvature k. As an application, Simons proved that if the square of the length of the second fundamental form A of a compact n-dimensional minimal submanifold M in a unit sphere S^{n+p} satisfies $|A|^2 < n/(2-1/p)$, then M is totally geodesic. Inspired by Simons' result, many pinching theorems for submanifolds in a sphere are proved.

For the study of submanifolds in a complex space form, Simons' type formula for generic submanifolds and CR submanifolds in a complex space form $M^m(c)$ of constant holomorphic sectional curvature c was given by Yano-Kon [7] under some additional conditions on the normal curvature and the mean curvature vector field. Using this formula, they proved some results of the pinching problem for the square of the length of the second fundamental form.

Recently, for a minimal CR submanifold of maximal CR dimension of a complex projective space CP^m of holomorphic sectional curvature 4, Djorić-Okumura [1] proved the following

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Theorem A. *Let M be an n-dimensional compact, minimal CR submanifold of maximal CR dimension of* $CP^{(n+k)/2}$ *. If the scalar curvature of M is greater than or equal to* $(n+2)(n-1)$, *then M is*

$$\pi\,(S^{m_1}(r_1)\times S^{m_2}(r_2)),\quad m_1+m_2=n+1,\quad r_i=(m_i/(n+1))^{1/2},$$

where $\pi : S^{2n+1}\longrightarrow CP^{(n+k)/2}$ *is the standard fibration.*

In this paper, we study the case that M is a compact minimal submanifold of CP^m with flat normal connection. For a vector field X tangent to M, we put PX the tangential part of JX, where J denotes the almost complex structure of CP^m. We prove

Theorem 1. *Let M be an n-dimensional compact minimal submanifold of* CP^m *with flat normal connection. If the scalar curvature r satisfies* $r \geq (n+2)|P|^2$, *then M is*

$$\pi(S^1(r_1)\times\cdots\times S^1(r_{n+1})),\qquad \sum_{i=1}^{n+1} r_i=(1/(n+1))^{1/2}$$

in some CP^n *in* CP^m, *or M is*

$$\pi\,(S^{m_1}(r_1)\times\cdots\times S^{m_k}(r_k)),\quad \sum_{i=1}^{k} m_i=n+1,\quad r_i=(m_i/(n+1))^{1/2},$$

where $m_1,\cdots,m_k$ *are odd numbers and* $2m=n+k-1$.

When M is a CR submanifold, $|P|^2$ is equal to the dimension of the holomorphic tangent space of the submanifold M.

2. Preliminaries.

Let $M^m(c)$ denote the complex space form of complex dimension m (real dimension $2m$) with constant holomorphic sectional curvature c. We denote by J the almost complex structure of $M^m(c)$. The Hermitian metric of $M^m(c)$ is denoted by G.

Let M be a real n-dimensional Riemannian manifold isometrically immersed in $M^m(c)$. We denote by g the Riemannian metric induced on M by G, and by p the codimension of M, that is, $p=2m-n$.

We denote by $T_x(M)$ and $T_x(M)^{\perp}$ the tangent space and the normal space of M, respectively.

Definition. A submanifold M of a Kählerian manifold $\tilde{M}$ is called a *CR submanifold of* $\tilde{M}$ if there exists a differentiable distribution $H : x \longrightarrow H_x \subset T_x(M)$ on M satisfying the following conditions:

(i) H is holomorphic, i. e., $JH_x=H_x$ for each $x\in M$, and

(ii) the complementary orthogonal distribution $H^{\perp}: x\longrightarrow H_x^{\perp}\subset T_x(M)$ is anti-invariant, i. e. $JH_x^{\perp}\subset T_x(M)^{\perp}$ for each $x\in M$.

We call H_x the holomorphic tangent space at x.

In the following, we put $\dim H_x = h$, $\dim H_x^{\perp} = q$ and codimension $M = p$. If $q = 0$ (resp. $h = 0$) for any $x \in M$, then the CR submanifold M is a holomorphic submanifold (resp. anti-invariant submanifold or totally real submanifold) of $\tilde{M}$. If $p = q$, then a CR submanifold M is called a *generic submanifold*. Any real hypersurface is a generic submanifold.

We denote by $\tilde{\nabla}$ the covariant differentiation in CP^m, and by ∇ the one in M determined by the induced metric. Then the *Gauss and Weingarten formulas* are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X, Y), \qquad \tilde{\nabla}_X V = -A_V X + D_X V$$

for any vector fields X and Y tangent to M and any vector field V normal to M, where D denotes the covariant differentiation with respect to the linear connection induced in the normal bundle $T(M)^{\perp}$ of M. A vector field V normal to M is said to be *parallel* if $D_X V = 0$ for any vector field X tangent to M. We call both A and B the *second fundamental form* of M and are related by $G(B(X,Y),V) = g(A_V X,Y)$. The second fundamental form A and B are symmetric.

For any vector field X tangent to M, we put

$$JX = PX + FX,$$

where PX is the tangential part of JX and FX the normal part of JX.Then P is an endomorphism on the tangent bundle $T(M)$ and F is a normal bundle valued 1-form on the tangent bundle $T(M)$.

For any vector field V normal to M, we put

$$JV = tV + fV,$$

where tV is the tangential part of JV and fV the normal part of JV. When M is CR submanifold, we see that $FP = 0, fF = 0$, $tf = 0$ and $Pt = 0$.

We define the covariant derivatives of P, F, t and f by $(\nabla_X P)Y = \nabla_X(PY) - P\nabla_X Y$, $(\nabla_X F)Y = D_X(FY) - F\nabla_X Y$, $(\nabla_X t)V = \nabla_X(tV) - tD_X V$ and $(\nabla_X f)V = D_X(fV) - fD_X V$, respectively. We then have

$$(\nabla_X P)Y = A_{FY}X + tB(X, Y), \quad (\nabla_X F)Y = -B(X, PY) + fB(X, Y),$$
$$(\nabla_X t)V = -PA_V X + A_{fV}X, \quad (\nabla_X f)V = -FA_V X - B(X, tV).$$

We denote by R the Riemannian curvature tensor field of M. Then the *equation of Gauss* is given by

$$\begin{aligned} R(X, Y)Z \;=\; & \frac{c}{4}\big(g(Y,Z)X - g(X,Z)Y + g(PY,Z)PX - g(PX,Z)PY \\ & -2g(PX,Y)PZ\big) + A_{B(Y,Z)}X - A_{B(X,Z)}Y \end{aligned}$$

for any X, Y and Z tangent to M.

We denote by S the Ricci tensor field of M. Then

$$S(X,Y) = \frac{c}{4}\Big((n-1)g(X,Y)+3g(PX,PY)\Big) + \sum_a \mathrm{tr}A_a g(A_aX,Y) - \sum_a g(A_a^2X,Y),$$

where A_a is the second fundamental form in the direction of v_a, $\{v_1,\cdots,v_p\}$ being an orthonormal basis for $T_x(M)^{\perp}$, and tr denotes the trace of an operator. From this the scalar curvature r of M is given by

$$r = \frac{c}{4}\Big((n-1)n - 3\mathrm{tr}P^2\Big) + \sum_a (\mathrm{tr}A_a)^2 - |A|^2.$$

We define the curvature tensor $R^{\perp}$ of the normal bundle of M by

$$R^{\perp}(X,Y)V = D_XD_YV - D_YD_XV - D_{[X,Y]}V.$$

Then we have the *equation of Ricci*

$$\begin{aligned} &g(R^{\perp}(X,Y)U,V) + g([A_V,A_U]X,Y) \\ &= \frac{c}{4}\Big(g(FY,U)g(FX,V) - g(FX,U)g(FY,V) + 2g(X,PY)g(fU,V)\Big). \end{aligned}$$

If $R^{\perp}$ vanishes identically, the normal connection of M is said to be *flat*. We can see that the normal connection of M is flat if and only if there exist locally p mutually orthogonal unit normal vector fields v_a such that each of the v_a's is parallel. If $R^{\perp}$ satisfies $R^{\perp}(X,Y)V = (c/2)g(X,PY)fV$, then the normal connection of M is said to be *semi-flat*.

We denote by CP^m a complex m-dimensional complex projective space of constant holomorphic sectional curvature 4.

Example. Let S^{n+k} be a $(n+k)$-dimensional unit sphere and N be a $(n+1)$ dimensional submanifold immersed in S^{n+k}. With respect to the Hopf fibration $\pi: S^{n+k} \longrightarrow CP^{(n+k+1)/2}$, we consider the following commutative diagram (cf. [2],[4],[7])

$$\begin{array}{ccc} N & \longrightarrow & S^{n+k} \\ \downarrow & & \downarrow \\ M & \longrightarrow & CP^{(n+k-1)/2} \end{array}.$$

We put

$$N = S^{m_1}(r_1) \times \cdots \times S^{m_k}(r_k), \quad n+1 = \sum_{i=1}^{k} m_i, \quad 1 = \sum_{i=1}^{k} r_i^2,$$

where $m_1,\cdots,m_k$ are odd numbers. We can see that $M=\pi(N)$ is a generic submanifold in $CP^{(n+k-1)/2}$ with flat normal connection. Moreover, M is a CR submanifold in CP^m $(m>(n+k-1)/2)$ with semi-flat normal connection and $\nabla f = 0$.

If $r_i = (m_i/(n+1))^{1/2}$ $(i=1,\cdots,k)$, then M is a generic minimal submanifold in $CP^{(n+k-1)/2}$.

For the proof of the main theorem, we use the following result.

Lemma 1 ([3]). *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$. Then*

$$\sum_a \mathrm{tr} A_{fa}^2 + \sum_a |[P, A_a]|^2 - 4\sum_a \mathrm{tr} A_a A_{fa} P \geq 0,$$

where A_{fa} denotes the second fundamental form in the direction of fv_a.

3. Submanifolds with flat normal connection.

In this section we study submanifolds of a complex space form with flat normal connection. First, we prove

Proposition 1. *Let M be an n-dimensional submanifold of $M^m(c)$, $c \neq 0$. If the normal connection of M is flat and $PA_a = A_aP$ for all a, then M is an anti-invariant submanifold or a generic submanifold of $M^m(c)$.*

Proof. We take an orthonormal basis $\{e_1, \cdots, e_h, tv_1 := e_{h+1}, \cdots, tv_q := e_n\}$ of $T_x(M)$, where $\{e_1, \cdots, e_h\}$ is an orthonormal basis of H_x and $\{v_1, \cdots, v_q\}$ is an orthonormal basis of $JH_x^\perp$. By the equation of Ricci, we have

$$\begin{aligned}
&\sum_{i=1}^{n} \left(g(A_{fU}A_U e_i, Pe_i) - g(A_U A_{fU} e_i, Pe_i) \right) \\
&= \frac{c}{4} \sum_i \left(g(FPe_i, U) g(Fe_i, fU) - g(Fe_i, U) g(FPe_i, fU) + 2g(e_i, P^2 e_i) g(fU, fU) \right) \\
&= -\frac{c}{2} \sum_i g(Pe_i, Pe_i) g(fU, fU).
\end{aligned}$$

If $PA_a = A_aP$ for all a, then

$$\begin{aligned}
&\mathrm{tr} A_U A_{fU} P - \mathrm{tr}\, A_{fU} A_U P = \mathrm{tr}\, A_U A_{fU} P - \mathrm{tr}\, A_U P A_{fU} \\
&= \mathrm{tr} A_U A_{fU} P - \mathrm{tr} A_U A_{fU} P = 0.
\end{aligned}$$

By $c \neq 0$, we have $\sum_i g(Pe_i, Pe_i) g(fU, fU) = 0$. If there exist $U \in T_x(M)^\perp$ such that $fU \neq 0$, then M is anti-invariant. If $fU = 0$ for all U, then M is a generic submanifold of $M^m(c)$.

When M is a CR submanifold of CP^m, Yano and Kon [7; p.237] proved the following

Theorem B. *Let M be an n-dimensional complete CR submanifold of CP^m with flat normal connection and parallel mean curvature vector. If $PA_V = A_VP$ for any vector field normal to M, then M is*

$$\pi\left(S^1(r_1) \times \cdots \times S^1(r_{n+1})\right), \quad \sum_{i=1}^{n+1} r_i^2 = 1$$

in some CP^n in CP^m, or M is

$$\pi\ (S^{m_1}\ (r_1)\ \times\cdots\times S^{mk}(r_k)),\quad \sum m_i = n+1,\quad \sum_{i=1}^{k} r_i^2 = 1,$$

where $m_1,\cdots, m_k$ *are odd numbers and* $2m = n + k - 1$.

Using Proposition 1, we can improve Theorem B.

Theorem 2. *Let M be an n-dimensional complete submanifold of* CP^m *with flat normal connection and parallel mean curvature vector. If* $PA_V = A_V P$ *for any vector field V normal to M, then M is an anti-invariant submanifold or*

$$\pi\ (S^1(r_1)\times\cdots\times S^1(r_{n+1})),\quad \sum_{i=1}^{n+1} r_i^2 = 1$$

in some CP^n *in* CP^m*, or M is*

$$\pi\ (S^{m_1}\ (r_1)\ \times\cdots\times S^{m_k}\ (r_k)),\quad \sum_{i=1}^{k} m_i = n+1, r_i^2 = 1,$$

where $m_1,\cdots, m_k$ *are odd numbers and* $2m = n + k - 1$.

4. Scalar curvature of a *CR* submanifold.

In this section, we prove the following theorem.

Theorem 3. *Let M be an n-dimensional compact minimal submanifold of* CP^m *with flat normal connection. If the scalar curvature r satisfies* $r \geq (n+2)|P|^2$*, then M is*

$$\pi\ (S^1(r_1)\ \times\cdots\times S^1(r_{n+1})),\quad \sum_{i=1}^{n+1} r_i = (1/(n+1))^{1/2}$$

in some CP^n *in* CP^m*, or M is*

$$\pi\ (S^{m_1}(r_1)\times\cdots\times S^{m_k}\ (r_k)),\quad \sum_{i=1}^{k} m_i = n+1,\quad r_i = (m_i/(n+1))^{1/2},$$

where $m_1,\cdots, m_k$ *are odd numbers and* $2m = n + k - 1$.

Proof. For any vector field X on a Riemannian manifold, we generally have the equation ([6])

$$\begin{aligned}&\mathrm{div}(\nabla_X X) - \mathrm{div}((\mathrm{div}X)X)\\ &= S(X,X) + \frac{1}{2}|L_X g|^2 - |\nabla X|^2 - (\mathrm{div}X)^2,\end{aligned}$$

where S denotes the Ricci tensor and $(L_X g)\ (Y,Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y)$.

Let M be an n-dimensional minimal submanifold in CP^m. Suppose that U is a parallel section of the normal bundle of M. From the equation of Gauss, we have

$$S(tU, tU) = (n-1)\,g(tU, tU) + 3g(PtU, PtU) - \sum_a g(A_a tU, A_a tU).$$

On the other hand, using the result that $(\nabla_X t)\,V = -PA_V X + A_{fV}X$ for any V normal to M, ∇_X

$(tU) = -PA_UX + A_{fU}X$. This implies $\mathrm{div}\,(tU) = \mathrm{tr}A_{fU} = 0$. We also have

$$|\nabla tU|^2 = \mathrm{tr}A_{fU}^2 + \mathrm{tr}A_U^2 - 2\mathrm{tr}A_UA_{fU}P - \sum_a g\,(A_Utv_a, A_Utv_a)\,,$$
$$|L_{tU}g|^2 = |[P, A_U]|^2 + 4\mathrm{tr}A_{fU}^2 - 8\mathrm{tr}A_UA_{fU}P.$$

Substituting these equations into the equation above, we have

$$\begin{aligned}\mathrm{div}\,(\nabla_{tU}tU) \;=\;& (n-1)\,g\,(tU, tU) + 3g\,(PtU, PtU) + \mathrm{tr}A_{fU}^2 - \mathrm{tr}A_U^2\\ & - 2\mathrm{tr}A_UA_{fU}P - \sum_a g\,(\mathrm{A}_a tU, \mathrm{A}_a tU)\\ & + \sum_a g\,(A_Utv_a, A_Utv_a) + \frac{1}{2}|[P, A_U]|^2\,.\end{aligned}$$

Since the normal connection of M is flat, we can choose an orthonormal frame $\{v_a\}$ of $T(M)^\perp$ such that $Dv_a = 0$ for all a. Hence we obtain

$$\begin{aligned}\sum_a \mathrm{div}\,(\nabla_{tv_a} tv_a) \;=\;& (\mathrm{n}-1)\,|t|^2 + 3|Pt|^2 + \sum_a \mathrm{tr}A_{fa}^2 - \sum_a \mathrm{tr}A_a^2\\ & - 2\sum_a \mathrm{tr}A_aA_{fa}P + \frac{1}{2}\sum_a |[P, A_a]|^2\,.\end{aligned}$$

Using Green's theorem, on a compact manifold M, $\int_M \mathrm{div}X *1 = 0$ for any $X \in T(M)$, where $*1$ is the volume element of M. Thus we have

$$\int_M \Big((n-1)\,|t|^2 + 3|Pt|^2 + \sum_a \mathrm{tr}A_{fa}^2 - \sum_a \mathrm{tr}A_a^2 - \sum_a 2\mathrm{tr}A_aA_{fa}P + \sum_a \frac{1}{2}|[P, A_a]|^2\Big) *1 = 0.$$

On the other hand, the scalar curvature r satisfies

$$r = (n-1)\,n + 3|P|^2 - \sum_a \mathrm{tr}A_a^2\,.$$

From these equations and Lemma 1, we see

$$\begin{aligned}0 \;=\;& \int_M \Big(r + (n-1)\,|t|^2 - 3|P|^2 - n\,(n-1) + 3|Pt|^2 + \frac{1}{2}\sum_a \mathrm{tr}A_{fa}^2\\ & + \frac{1}{2}\big(\sum_a \mathrm{tr}A_{fa}^2 + \sum_a |[P, A_a]|^2 - 4\sum_a \mathrm{tr}A_aA_{fa}P\big)\Big) *1\\ \geq\;& \int_M (r - (n+2)\,|P|^2) *1.\end{aligned}$$

Here we used $|P|^2 + |t|^2 = n$. Since $r - (n+2)\,|P|^2 \geq 0$, the right-hand side of this equation is non-negative. Hence we obtain

$$|Pt|^2 = 0,$$
$$\frac{1}{2}\sum_a \mathrm{tr}A_{fa}^2 = 0,$$
$$\sum_a \mathrm{tr}A_{fa}^2 + \sum_a |[P, A_a]|^2 - 4\sum_a \mathrm{tr}A_aA_{fa}P = 0.$$

From these equations, we see that M is a CR submanifold which satisfy $A_{fa} = 0$ and $A_aP = PA_a$ for any v_a. Since M is minimal, using Theorem 1, we have our theorem.

As a consequence of the theorem, we have

Corollary 1. Let M be an n-dimensional compact minimal CR submanifold of CP^m with flat normal connection. If the scalar curvature r satisfies $r \geq h(n+2)$, where h denotes the dimension of the holomorphic tangent space of M, then M is

$$\pi\ (S^1(r_1) \times \cdots \times S^1(r_{n+1})), \quad \sum_{i=1}^{n+1} r_i = (1/(n+1))^{1/2}$$

in some CP^n in CP^m, or M is

$$\pi\ (S^{m_1}(r_1) \times \cdots \times S^{m_k}(r_k)), \quad \sum_{i=1}^{k} m_i = n+1, \quad r_i = (m_i/(n+1))^{1/2},$$

where $m_1, \cdots, m_k$ are odd numbers and $2m = n + k - 1$.

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