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# Ricci solutions and real hypersurfaces

# リッチ解と実超曲面

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**Abstract**

We study the expression similar to the equation of Ricci solution. As applications we give conditions for real hypersurfaces of a complex space form to be some Hopf hypersurface.

**Key words**: real hypersurface, shape operator, complex space form, Ricci solution

**Introduction.**

In [2], Cho and Kimura studied on Ricci solutions of real hypersurfaces in a non-flat complex space form. They proved that a real hypersurface $M$ in a non-flat complex space form $\bar{M}^n(c)$ with $c \neq 0$ does not admit a Ricci solution whose solution vector field is the structure vector field $\xi$. In this context, they define so called $\eta$*-Ricci solution* $(\eta, g)$, which satisfies

$$\frac{1}{2} L_\xi g + S - \lambda g - \mu \eta \otimes \eta = 0$$

for constants $\lambda$, $\mu$, and classified $\eta$-Ricci solution real hypersurfaces in a non-flat complex space form.

In this paper, we study a generalized equation of the above, that is,

$$(L_\xi g)(X, Y) + g(TX, Y) = 0,$$

where $T$ is a symmetric (1, 1) tensor field which satisfies $g(TAX, Y) = g(ATX, Y)$ for any vectors $X, Y$ in the holomorphic subspace $H_x(M)$ of the tangent space $T_x(M)$ of $M$. Then we prove that $M$ is a real hypersurface with $\phi A = A \phi$, where $A$ is the shape operator and $\phi$ is the induced almost contact structure of $M$.

**1. Preliminaries.**

Let $\bar{M}$ be a complex $n$-dimensional Kaehler manifold. We denote by $J$ the almost complex

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structure of $\bar{M}$. The Hermitian metric of $\bar{M}$ will be denoted by $G$.

Let $M$ be a real $(2n-1)$-dimensional hypersurface immersed in $\bar{M}$. We denote by $g$ the Riemannian metric induced on $M$ from $G$. We take the unit normal vector field $N$ of $M$ in $\bar{M}$. For any vector field $X$ tangent to $M$, we define $\phi$, $\eta$ and $\xi$ by

$$JX=\phi X+\eta(X)N, \qquad JN=-\xi,$$

where $\phi X$ is the tangential part of $JX$, $\phi$ is a tensor field of type $(1,1)$, $\eta$ is a 1-form, and $\xi$ is the unit vector field on $M$. Then they satisfy

$$\phi^2 X=-X+\eta(X)\xi, \qquad \phi\xi=0, \qquad \eta(\phi X)=0$$

for any vector field $X$ tangent to $M$. Moreover, we have

$$g(\phi X, Y)+g(X,\phi Y)=0, \quad \eta(X)=g(X,\xi),$$
$$g(\phi X, \phi Y)=g(X,Y)-\eta(X)\eta(Y).$$

Thus $(\phi, \xi, \eta, g)$ defines an almost contact metric structure on $M$.

We denote by $\tilde{\nabla}$ the operator of covariant differentiation in $\bar{M}$, and by $\nabla$ the one in $M$ determined by the induced metric. Then the *Gauss and Weingarten formulas* are given respectively by

$$\tilde{\nabla}_X Y=\nabla_X Y+g(AX,Y)N, \qquad \tilde{\nabla}_X N=-AX,$$

for any vector fields $X$ and $Y$ tangent to $M$. We call $A$ the *shape operator* of $M$.

For the contact metric structure on $M$ we have

$$\nabla_X\xi=\phi\, \mathrm{AX}, \qquad (\nabla_X\phi)Y=\eta(Y)AX-g(AX,Y)\xi.$$

As an ambient manifold we take $\bar{M}^n(c)$ the complex space form of complex dimension $n$ with constant holomorphic sectional curvature $4c$.

We denote by $R$ the Riemannian curvature tensor field of $M$. Then the *equation of Gauss* is given by

$$\begin{aligned} R(X,Y)Z &= c\,\{g(Y,Z)X-g(X,Z)Y+g(\phi Y,Z)\phi X \\ &\quad -g(\phi X,Z)\,\phi Y-2g(\phi X,Y)\,\phi Z\} \\ &\quad +g(AY,Z)AX-g(AX,Z)AY, \end{aligned}$$

and the *equation of Codazzi* by

$$(\nabla_X A)Y-(\nabla_Y A)X=c\{\eta(X)\phi Y-\eta(Y)\phi X-2g(\phi X,Y)\xi\}.$$

From the equation of Gauss, the Ricci tensor $S$ of $M$ is given by

$$\begin{aligned} S(X,Y) &= (2n+1)cg(X,Y)-3c\eta(X)\eta(Y) \\ &\quad +\mathrm{Tr}A\,g(AX,Y)-g(AX,AY), \end{aligned}$$

where $\mathrm{Tr}A$ is the trace of $A$.

If the shape operator $A$ of $M$ satisfies $A\xi=\alpha\xi$, $\alpha$ being a function, then $M$ is called a *Hopf hypersurface*.

If the shape operator $A$ of $M$ is of the form $AX = aX + b\eta(X)\xi$ for some functions $a$ and $b$, then $M$ is said to be *totally $\eta$-umbilical* (see [8]). It is well known that if $M$ is a totally $\eta$-umbilical real hypersurface of a complex space form $\bar{M}^n(c)$, $c \neq 0$, $n \geq 2$, then $M$ has two constant principal curvatures.

If the Ricci tensor $S$ of $M$ is of the form $S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$ for some functions $a$ and $b$, then $M$ is said to be *pseudo-Einstein* (see [3]). In these cases, $a$ and $b$ are constant.

## 2. Theorem.

First we prove

**Theorem 1.** *Let $M$ be a real hypersurface of a Kaehler manifold $\bar{M}$. If we have*

$$(L_\xi g)(X, Y) + g(TX, Y) = 0,$$

*where $T$ is a symmetric (1, 1) tensor which satisfies $g(TAX, Y) = g(ATX, Y)$ for any $X, Y \in H_x(M)$, then $M$ is a Hopf hypersurface with $\phi A = A\phi$.*

*Proof.* Taking an orthonormal basis $\{e_1, \cdots, e_{2n-2}, e_{2n-1} = \xi\}$ of $T_x(M)$, we have

$$\begin{aligned} 0 &= \sum_i (g((\phi A - A\phi)e_i, A\phi e_i) + g(Te_i, A\phi e_i)) \\ &= \frac{1}{4}|[\phi, A]|^2 + \mathrm{Tr}(\phi AT) = \frac{1}{4}|[\phi, A]|^2. \end{aligned}$$

**Theorem 2** ([**2**]). *Let $M$ be a real hypersurface of a complex space form $M^n(c)$. If*

$$\frac{1}{2}L_\xi g + S - \lambda g - \mu\eta \otimes \eta = 0,$$

*then $M$ is a pseudo-Einstein real hypersurface with $\phi A = A\phi$.*

*Proof.* We *take* $T$ in Theorem 1 by

$$\begin{aligned} g(TX, Y) &= S(X, Y) - \lambda g(X, Y) - \mu\eta(X)\eta(Y) \\ &= (2n+1)cg(X, Y) - 3c\eta(X)\eta(Y) \\ &\quad + \mathrm{Tr}Ag(AX, Y) - g(AX, AY) - \lambda g(X, Y) - \mu\eta(X)\eta(Y). \end{aligned}$$

Then, $T$ is symmetric and $g(TAX, Y) = g(ATX, Y)$ for any $X, Y \in H_x(M)$. Therefore, by Theorem 1, $L_\xi g = 0$ and $M$ is a pseudo Einstein real hypersurface.

**Theorem 3**. *Let $M$ be a real hypersurface of a Kaehler manifold $\bar{M}$. If*

$$(L_\xi g)(X, Y) + \alpha g(AX, Y) - \lambda g(X, Y) - \mu\eta(X)\eta(Y) = 0,$$

*where $\alpha \neq 0$, then $M$ is a totally $\eta$-umbilical real hypersurface with $\phi A = A\phi$.*

*Proof.* We take $T$ in Theorem 1 by

$$g(TX, Y) = \alpha g(AX, Y) - \lambda g(X, Y) - \mu\eta(X)\eta(Y).$$

Then, $T$ is symmetric and $g(TAX, Y) = g(ATX, Y)$ for any $X, Y \in H_x(M)$. Therefore $L_\xi g = 0$ and $M$ is a totally $\eta$-umbilical real hypersurface.

**Remark 1**. Real hypersurfaces $M$ of a complex space form with $\phi A = A\phi$ were stidied by many authors (cf. [1], [4], [6, 7]).

**Remark 2**. A *Ricci solution* is defined by

$$\frac{1}{2} L_V g + S - \lambda g = 0,$$

where $V$ is a vector field (the potential vector field) and $\lambda$ a constant on $M$. If a real hypersurface $M$ of a non-flat complex space form admits a Ricci solution for $V = \xi$, then $\phi A = A\phi$ and $M$ is an Einstein real hypersurface. But, it is well known that there does not exists Einstein real hypersurface of a non-flat complex spae form (cf. [3]). Thus $M$ does not admits a Ricci solution for $V = \xi$ ([2]).

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