Einstein hypersurfaces in an odd-dimensional sphere

奇数次元球面のアインシュタイン超曲面

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Abstract

We study a hypersurface immersed in an odd-dimensional sphere with the induced structure from the contact metric structure. We prove that if a hypersurface of an odd-dimensional sphere admits a Ricci soliton with the potential vector field constructed by the unit normal vector field, then M is an Einstein hypersurface.

Key words: Ricci soliton, hypersurface, contact metric structure

1. Introduction

In [1], Cho and Kimura studied on Ricci solitons of real hypersurfaces in a non-flat complex space form. They proved that a real hypersurface M in a non-flat complex space form $\bar{M}^n(c)$ with $c \neq 0$ does not admit a Ricci soliton whose soliton vector field is the structure vector field ξ. In this context, they define so called *η-Ricci soliton* (η, g), which satisfies

$$\frac{1}{2} L_{\xi} g + S - kg - \mu\eta \otimes \eta = 0$$

for constants k, μ, and classified η-Ricci soliton real hypersurfaces in a non-flat complex space form.

In this paper, we study a hypersurface M immersed in a unit sphere S^{2n+1} with contact metric structure (ϕ, ξ, η, g) and the Ricci soliton on M

$$\frac{1}{2} L_U g + S - kg = 0$$

where U is a vector field defined by $U = \phi C$, C being the unit normal of M in S^{2n+1}.

We prove that if a hypersurface M of an odd-dimensional sphere S^{2n+1} admits a Ricci soliton with the potential vector fined U constructed by the unit normal vector field C, then M is an Einstein hypersurface.

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2. Preliminaries

Let S^{2n+1} be a (2n+1)-dimensional unit sphere of constant curvature 1. It is well known that S^{2n+1} admits the standard Sasakian structure (normal contact metric structure) (ϕ, ξ, η, g). Then they satisfy (cf. [4])

$$\phi^2 X = -X + \eta(X)\xi, \qquad \phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad \eta(\xi) = 1,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad \eta(X) = g(X, \xi)$$

for any vector fields X and Y on S^{2n+1}.

We denote by $\bar{\nabla}$ the operator of covariant differentiation with respect to g. Then

$$\bar{\nabla}_X \xi = \phi X, \qquad (\bar{\nabla}_X \phi)Y = \eta(Y)X - g(X, Y)\xi.$$

Let M be a $2n$-dimensional hypersurface immersed in S^{2n+1}. We denote by the same g the induced metric tensor field of M. Let C be a unit normal of M in S^{2n+1}. For any vector field X tangent to M we put

$$\phi X = fX + u(X)C, \qquad \xi = V + \lambda C, \qquad \phi C = -U,$$
$$v(X) = \eta(X), \qquad \lambda = \eta(C) = g(\xi, C),$$

where f is a tensor field of type (1,1), u, v 1-forms, U, V vector fields and λ a scalar function on M. Then (cf. [5])

$$f^2 X = -X + u(X)U + v(X)V, \qquad u(fX) = \lambda v(X), \qquad v(fX) = -\lambda u(X),$$
$$fU = -\lambda V, \qquad fV = \lambda U, \qquad u(V) = 0, \qquad v(U) = 0,$$
$$u(U) = 1 - \lambda^2, \qquad v(V) = 1 - \lambda^2.$$

Moreover, we have

$$g(U, X) = u(X), \qquad g(V, X) = v(X), \qquad g(fX, Y) = -g(X, fY),$$
$$g(fX, fY) = g(X, Y) - u(X)u(Y) - v(X)v(Y).$$

For any vector fields X and Y tangent to M, we have the Gauss and Weingarten formulas

$$\bar{\nabla}_X Y = \nabla_X Y + g(AX, Y)C, \qquad \bar{\nabla}_X C = -AX,$$

where ∇ denotes the operator of covariant differentiation in M and A the shape operator of M. Then we have

$$\nabla_X V = fX + \lambda AX, \qquad \nabla_X U = -\lambda X + fAX,$$
$$X\lambda = u(X) - g(AX, V).$$

We denote by R the Riemannian curvature tensor field of M. Then the equation of Gauss is given by

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(AY,Z)AX - g(AX,Z)AY,$$

and the equation of Codazzi is given by

$$(\nabla_X A)Y - (\nabla_Y A)X = 0.$$

We denote by S the Ricci tensor of M. Then

$$S(X,Y) = (2n-1)g(X,Y) + \mathrm{Tr}A g(AX,Y) - g(A^2X,Y).$$

We prepare the basic properties for λ.

Lemma 1. *We have $\lambda^2 \neq 1$ almost everywhere on M.*

Proof. If $\lambda^2 = 1$, then the structure vector field ξ is normal to M. Then $\bar{\nabla}_X \xi = -AX = \phi X$. Since A is symmetric and ϕ is skew-symmetric, we see $\phi X = 0$. This is a contradiction.

Lemma 2. *If $Af = fA$ and λ is constant, then $\lambda = 0$.*

Proof. If λ is constant, then $u(X) = g(AX, V)$ and hence $AV = U$. Then we have

$$0 = g(fAU,U) - g(AfU,U) = 2\lambda g(AV,U) = 2\lambda u(U) = 2\lambda(1-\lambda^2).$$

Using Lemma 1, we have $\lambda = 0$.

3. Ricci solitons on hypersurfaces

We denote by L_W the Lie differentiation with respect to a vector field W on a Riemannian manifold (M,g). A Ricci soliton is defined on (M,g) by

$$\frac{1}{2}(L_W g)(X,Y) + S(X,Y) - kg(X,Y) = 0,$$

where W is a vector field (the potential vector field) and k a constant on M.

Lemma 3. Let *M be a hypersurface of S^{2n+1}. If M admits a Ricci soliton with the potential vector field U, then we have $Af = fA$.*

Proof. Let $\{e_1,\cdots,e_{2n}\}$ be an orthonormal basis of M. Since

$$(L_U g)(X,Y) = g(\nabla_X U, Y) + g(\nabla_Y U, X),$$

we have

$$\sum\left(\frac{1}{2}(L_U g)(e_i, Afe_i) - S(e_i, Afe_i) - kg(e_i, Afe_i)\right)$$
$$= \frac{1}{2}\sum\left(g(\nabla_{e_i} U, Afe_i) + g(\nabla_{Afe_i} U, e_i)\right)$$

$$
\begin{aligned}
&-\sum(2n-1)\,g(e_i, Afe_i) + \mathrm{Tr}A\sum g(Ae_i, Afe_i)\\
&-\sum g(A^2e_i, Afe_i) - k\sum g(e_i, Afe_i)\\
&=\frac{1}{2}\sum g(-\lambda e_i + fAe_i, Afe_i) + \frac{1}{2}\sum g(-\lambda Afe_i + fA^2fe_i, e_i)\\
&=\frac{1}{2}\sum\left(g(fAe_i, Afe_i) - g(Afe_i, Afe_i)\right)\\
&=-\frac{1}{4}|[f, A]|^2 = 0.
\end{aligned}
$$

This means $Af = fA$.

Theorem 1. *Let M be a hypersurfaces of S^{2n+1}, $n > 1$. If M admits a Ricci soliton with the potential vector field U, then M is an Einstein hypersurface and locally congruent to*

$$
S^p\left(\frac{2n-2}{p-1}\right) \times S^{2n-p}\left(\frac{2n-2}{2n-p-1}\right),
$$

where $p\,(1 < p < 2n-1)$ is an odd number and $S^p(r)$ denotes a p-dimensional sphere of constant curvature r.

Proof. Form Lemma 3.1, we have $Af = fA$. Hence we have

$$
\begin{aligned}
&(L_Ug)(X, Y)\\
&= g(\nabla_X U, Y) + g(\nabla_Y U, X)\\
&= g(fAX, Y) - \lambda g(X, Y) + g(fAY, X)\\
&\quad -\lambda g(Y, X)\\
&= g((fA - Af)X, Y) - 2\lambda g(X, Y)\\
&= -2\lambda g(X, Y).
\end{aligned}
$$

By the assumption,

$$
\begin{aligned}
&\frac{1}{2}(L_Ug)(X, Y) + S(X, Y) - kg(X, Y)\\
&\quad = S(X, Y) - (\lambda + k)\, g(X, Y) = 0.
\end{aligned}
$$

Therefore M is an Einstein hypersurface. If dim $M \geq 3$, then $\lambda + k$ is constant. Since k is constant, λ is also a constant. Then, by Lemma 2, $\lambda = 0$. Hence the structure vector field ξ is tangent to M. Moreover, we have

$$
fU = 0, \qquad fV = 0, \qquad u(U) = 1, \qquad v(V) = 1,
$$
$$
\nabla_X V = fX, \qquad \nabla_X U = fAX, \qquad AV = U.
$$

Since $fU = 0$, we obtain $fAU = AfU = 0$ and hence

$$
AU = \alpha U + V, \qquad \alpha = u(AU).
$$

By the equation of Codazzi,

$$
\begin{aligned}
&g((\nabla_X A)Y, U) - g((\nabla_Y A)X, U)\\
&= g(Y, (\nabla_X A)U) - g(X, (\nabla_Y A)U)\\
&= g(Y, \nabla_X AU) - g(Y, A\nabla_X U) - g(X, \nabla_Y AU) + g(X, A\nabla_Y U)
\end{aligned}
$$

$$\begin{aligned} &= \alpha g(Y, fAX) + g(Y, fX) - g(Y, AfAX) \\ &\quad - \alpha g(X, fAY) - g(X, fY) + g(X, AfAY) \\ &= \alpha g((fA + Af)X, Y) + 2g(fX, Y) - 2g(AfAX, Y) \\ &= 2\alpha g(fAX, Y) + 2g(fX, Y) - 2g(AfAX, Y) = 0 \end{aligned}$$

for any X, Y orthogonal to U and V. Consequently, we have

$$0 = \alpha g(fAX, fX) + g(fX, fX) - g(fAX, AfX).$$

From $fA = Af$, if $AX = aX$, then $AfX = fAX = afX$. Let X satisfies $AX = aX$ and $g(X, U) = g(X, V) = 0$. Then we have

$$a^2 - \alpha a - 1 = 0.$$

Therefore we can take an orthonormal basis of M such that the shape operator A can be represented as

$$A = \left(\begin{array}{ccccccc|cc} a & & & & & & & & \\ & \ddots & & & & & & & \\ & & a & & & & & & \\ & & & b & & & & & \\ & & & & \ddots & & & & \\ & & & & & & b & & \\ \hline & & & & & & & 0 & 1 \\ & & & & & & & 1 & \alpha \end{array}\right)$$

where $ab = -1$ and $a + b = \alpha$. The eigenvalue x of the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & \alpha \end{pmatrix}$$

satisfies

$$x^2 - \alpha x - 1 = 0.$$

Therefore A has two eigenvalues a and b. We put

$$\mathrm{Tr}A = pa + qb, \qquad p + q = 2n,$$

where p is odd. If $AX = aX$ and $AY = bY$, then we have

$$S(X, X) = (2n - 1) + \mathrm{Tr}A \cdot a - a^2,$$
$$S(Y, Y) = (2n - 1) + \mathrm{Tr}A \cdot b - b^2.$$

Since M is Einstein, we have

$$(\mathrm{Tr}A - a - b)(a - b) = 0.$$

By $ab = -1$, we have $a \neq b$. Hence

$$0 = (p - 1)a + (q - 1)b = (p - 1)a + (q - 1)\left(-\frac{1}{a}\right).$$

Thus we obtain

$$a^2 = \frac{q - 1}{p - 1} = \frac{2n - p - 1}{p - 1}, \qquad b^2 = \frac{p - 1}{2n - p - 1}.$$

Therefore a and b are constant. We consider the distributions defined by

$$T_a(x) = \{X \mid AX = aX\}, \qquad T_b(x) = \{Y \mid AX = bY\}.$$

Then T_a and T_b are parallel distribution and maximal integral manifolds are totally umbilical submanifolds with constant curvatures (see [3]). That is, the maximal integral manifold M_1 of T_a is of constant curvatures

$$1 + \frac{2n-p-1}{p-1} = \frac{2n-2}{p-1}$$

and is totally umbilical in S^{2n+1}, and the maximal integral manifold M_2 of T_b is totally umbilical in S^{2n+1} and is of constant curvature

$$1 + \frac{p-1}{2n-p-1} = \frac{2n-2}{2n-p-1}.$$

Therefore, M is locally isometric to the product of spheres

$$Sp\left(\frac{2n-2}{p-1}\right) \times S^{2n-p}\left(\frac{2n-2}{2n-p-1}\right),$$

where p is an odd number such that $1 < p < 2n-1$.

Next we consider the condition that

$$\frac{1}{2} L_U g + S - kg = 0$$

under the assumption that k is a function. First, we prepare the following lemma.

Lemma 4. *If $fA = Af$, then $\lambda = 0$ or $U\lambda = 1 - \lambda^2$.*

Proof. Since we have $fU = -\lambda V$, $fV = \lambda U$ and $u(U) = v(V) = 1 - \lambda^2$, we have

$$fAU = AfU = -\lambda AV.$$

Thus we obtain

$$g(fAU, U) = -g(AU, fU) = \lambda g(AU, V).$$

On the other hand, we have

$$g(fAU, U) = g(AfU, U) = -\lambda g(AV, U).$$

From these equation, we see that $\lambda g(AU, V) = 0$. Since $X\lambda = u(X) - g(AX, V)$, we have

$$U\lambda = u(U) - g(AU, V) = (1 - \lambda^2) - g(AU, V).$$

Thus we obtain

$$\lambda(U\lambda) = \lambda(1 - \lambda^2) = 0.$$

This proves our assertion.

Theorem 2. *Let M be a hypersurface of S^{2n+1}, $n > 1$. If M satisfies*

$$\frac{1}{2} L_U g + S - kg = 0,$$

where k is a function on M, then M is locally isometric to

$$S^p\left(\frac{2n-2}{p-1}\right) \times S^{2n-p}\left(\frac{2n-2}{2n-p-1}\right),$$

where $p\,(1 < p < 2n-1)$ *is an odd number, or* $S^{2n}(1+\alpha^2)$, $\alpha = v(Av)/(1-\lambda^2)$.

Proof. From Lemma 4, we have $\lambda = 0$ or $U\lambda = 1 - \lambda^2$. When $\lambda = 0$, then the proof of Theorem 1 implies that M is congruent to

$$S^p\left(\frac{2n-2}{p-1}\right) \times S^{2n-p}\left(\frac{2n-2}{2n-p-1}\right),$$

where p is an odd number.

Next we consider the case that $U\lambda = 1 - \lambda^2$. We notice $1 - \lambda^2 \neq 0$. Then λ is not constant, and hence $\lambda \neq 0$. Then we have $g(AU, V) = 0$. Since $fA = Af$, we see, by $fV = \lambda U$,

$$fAV - \lambda AU = 0,$$

so that

$$\begin{aligned} 0 &= f^2AV - \lambda fAU \\ &= -AV + u(AV)U + v(AV)V + \lambda^2 AV \\ &= -AV + v(AV)V + \lambda^2 AV. \end{aligned}$$

Then we have

$$AV = \alpha V, \qquad \alpha = \frac{v(AV)}{1-\lambda^2}.$$

On the other hand, from $fAU + AfU = 0$, we see $fAU = -\lambda AV$. This implies

$$g(fAU, V) = -g(AU, fV) = -\lambda g(AU, U) = -\lambda g(AV, V).$$

Hence we have $u(AU) = v(AV)$. From this, we have also $AU = \alpha U$.

Moreover, we have

$$\begin{aligned} (\nabla_X A)V &= \nabla_X AV - A\nabla_X V \\ &= (X\alpha)V + \alpha\,(fX + \lambda AX) - A(fX + \lambda AX). \end{aligned}$$

So we obtain

$$\begin{aligned} g((\nabla_X A)V, Y) &= (X\alpha)v(Y) + \alpha g(fX, Y) + \alpha\lambda g(AX, Y) \\ &\quad - g(AfX, Y) - \lambda g(A^2X, Y), \\ g((\nabla_Y A))V, X) &= (Y\alpha)v(X) + \alpha g(fY, X) + \alpha\lambda g(AY, X) \\ &\quad - g(AfY, X) - \lambda g(A^2Y, X). \end{aligned}$$

By the equation of Codazzi, we have

$$\begin{aligned} 0 &= g((\nabla_X A)V, Y) - g((\nabla_Y A)V, X) \\ &= (X\alpha)v(Y) - (Y\alpha)v(X) + 2\alpha g(fX, Y) - 2g(fAX, Y). \end{aligned}$$

Putting $Y = V$, we get

$$0 = (X\alpha)(1-\lambda^2) - (V\alpha)v(X) - 2\alpha\lambda u(X) + 2\lambda u(AX).$$

Since we have $u(AX) = g(U, AX) = \alpha u(X)$,

$$0 = (X\alpha)(1-\lambda^2) - (V\alpha)v(X),$$
$$0 = (Y\alpha)(1-\lambda^2) - (V\alpha)v(Y).$$

So we have

$$X\alpha = \frac{(V\alpha)v(X)}{1-\lambda^2}, \qquad Y\alpha = \frac{(V\alpha)v(Y)}{1-\lambda^2}.$$

Substituting these into the equation above, we have

$$fAX = \alpha fX$$

for X orthogonal to U and V. Thus we have $AX = \alpha X$. Then M is totally umbilical and is of constant curvature $1 + \alpha^2$.

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