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Infinite Product Representations for Vignéras' Multiple Gamma Functions

（Vignéras の多重ガンマ関数の無限積表示）

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Abstract　Gauss' and Euler's type infinite product representations for Vignéras multiple gamma function are presented. As an application of the representations, a multiplication formula for the function is derived.

Key words: multiple gamma function, Gauss product representation, Euler product representation, multiplication formula

1. Introduction

In a series of papers [2, 3, 4, 5], Barnes introduced multiple gamma functions associated with a certain generalization of the Hurwitz zeta function. In relevant with a special case of Barnes' function, Vignéras [15] introduced her multiple gamma functions $G_r(z)$ $(r \in \mathbb{Z}_{\geq 0})$ as a sequence of meromorphic functions uniquely determined by the following relations:

$$\begin{aligned} &\text{(i) } G_0(z) = z, \quad \text{(ii) } G_r(1) = 1, \quad \text{(iii) } G_r(z+1) = G_{r-1}(z) G_r(z) \\ &\text{(iv) } \frac{d^{r+1}}{dz^{r+1}} \log G_r(z+1) \geq 0 \quad \text{for } z \geq 0. \end{aligned} \tag{1}$$

This formulation can be considered as a generalization of the Bohr-Morellup theorem. For example, $G_1(z)$ is the celebrated Euler gamma function $\Gamma(z)$ (*cf.* Artin [1], Whittaker-Watson [16].). $G_2(z)$ is G-function introduced in Barnes [2].

In this paper, we present two types of infinite product representations of Vignéras' multiple gamma function, which can be considered as a generalization of the Gauss and the Euler product formula of Euler's gamma function

$$\Gamma(z+1) = \lim_{N\to\infty} \frac{N!}{(z+1)(z+2)\cdots(z+N)} (N+1)^z \tag{2}$$

$$= \prod_{n=1}^{\infty} \left[\left(1+\frac{z}{n}\right)^{-1} \left(1+\frac{1}{n}\right)^{z} \right] \tag{3}$$

(*cf.* Artin [1], Whittaker-Watson [16]). Our main theorem is stated as follows: If z is not negative

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integer, the multiple gamma function $G_r(z)$ is represented as

$$G_r(z+1) = \lim_{N\to\infty}\left[\prod_{n=1}^{N}\frac{G_{r-1}(n)}{G_{r-1}(z+n)}\prod_{k=0}^{r-1}G_k(N+1)^{\binom{z}{r-k}}\right] \tag{4}$$

$$= \prod_{n=1}^{\infty}\left[\frac{G_{r-1}(n)}{G_{r-1}(z+n)}\prod_{k=0}^{r-1}\left(\frac{G_k(n+1)}{G_k(n)}\right)^{\binom{z}{r-k}}\right]. \tag{5}$$

In the case when $r = 1$, these formulas coincide with (2) and (3). We can find the representation for $G_2(z)$ in Jackson [6]. It should be noted that infinite product formula of these types for a q-analogue of the multiple gamma function were already obtained in [12]. However, in contrast to simplicity in q-case, some delicate techniques are necessary to deal with infinite products of Vignéras' function. We verify (4) and (5) in section 1. The point is to apply an asymptotic expansion in [13] to estimations for products of Vignéras' functions.

In section 2, as an application of infinite product representations, we derive a multiplication formula for Vignéras' multiple gamma function, which can be regarded as a generalization of the well known formula

$$\prod_{m=0}^{p}\Gamma\left(\frac{z+m}{p}\right) = \frac{(2\pi)^{\frac{p-1}{2}}}{p^{z-\frac{1}{2}}}\Gamma(z) \tag{6}$$

for Euler's gamma function (*cf.* Artin [1], Whittaker-Watson [16]). It is described as follows:

$$\prod_{q_1,\ q_2,\cdots q_r=0}^{p-1} G_r\left(\frac{z+q_1+\cdots+q_r}{p}\right) = \frac{e^{\phi_r(z)}}{p^{\psi_r(z)}}G(z)$$

It might be seem that formula of this type can be guessed easily from (1). However, it is not easy to determine explicit forms of $\phi_r(z)$ and $\psi_r(z)$. The reason why we can do it is usefulness of our representations (4).

For simplicity, we call Vignéras multiple gamma function only "multiple gamma function" in the following sections.

Notations: In this paper, we use notation $B_r(z)$ for the Bernoulli polynomial defined by the generating function

$$\sum_{r=0}^{\infty}B_r(z)t^r = \frac{te^t}{1-e^t},$$

and B_r for the Bernoulli number defined as $B_r := B_r(0)$. We introduce the Stirling number ${}_rS_j$ of the 1 st kind by

$$t(t-1)\cdots(t-r+1) = \sum_{j=0}^{r}{}_rS_j t^j.$$

The notation $\zeta(s)$ is used to refer to the Riemann zeta function defined as the series $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$ and its analytical continuation. $\zeta'(s)$ is the first derivative of $\zeta(s)$ defined by $\zeta'(s) := \frac{d}{ds}\zeta(s)$.

2 Innite product representations

As mentioned in introduction, our main theorem is described as follows:

Theorem 2.1 *If z is not negative integer and is included in any nite region of complex plane, the multiple gamma function* $G_r(z)$ *is represented as*

$$G_r(z+1) = \lim_{N\to\infty}\left[\prod_{n=1}^{N}\frac{G_{r-1}(n)}{G_{r-1}(z+n)}\prod_{k=0}^{r-1}G_k(N+1)^{\binom{z}{r-k}}\right] \tag{7}$$

$$= \prod_{n=1}^{\infty}\left[\frac{G_{r-1}(n)}{G_{r-1}(z+n)}\prod_{k=0}^{r-1}\left(\frac{G_k(n+1)}{G_k(n)}\right)^{\binom{z}{r-k}}\right]. \tag{8}$$

Proof. From the Gauss product representation (7), the Euler product representation (8) follows immediately. So, we give a proof of (7) in this section. We apply an asymptotic expansion for $G_r(z)$, which was firstly appeared in [13].

Theorem 2.2 (**Ueno-Nishizawa**) *Let us put* $0<\delta<\pi$, *then, as* $|z|\to\infty$ *in the sector* $\{z\in\mathbb{C} \mid |\arg z|<\pi-\delta\}$,

$$\begin{aligned}\log G_r(z+1) = &\left\{\binom{z+1}{r}+\sum_{j=0}^{r-1}\frac{B_{j+1}}{j+1}G_{r,j}(z)\right\}\log(z+1)- \\ &-\sum_{j=0}^{r-1}G_{r,j}(z)\frac{(z+1)^{j+1}}{(j+1)^2}-\sum_{j=0}^{r-1}G_{r,j}(z)\zeta'(-j)+O(z^{-1}).\end{aligned} \tag{9}$$

where a polynomial $G_{r,j}(z)$ *is dened by the generating function*

$$\binom{z-u}{r-1} =: \sum_{j=0}^{r-1}G_{r,j}(z)u^j \quad (r=0\cdots r-1), \qquad G_{r,j}(z)=0, \quad (j\ge r).$$

In our proof, the following lemma is useful:

Lemma 2.3 *For arbitrary* $x, y\in\mathbb{C}$,

$$\text{(i)}\quad \sum_{k=0}^{r}\binom{x}{r-k}\binom{y}{k}=\binom{x+y}{r}, \qquad \text{(ii)}\quad \sum_{k=0}^{r}\binom{x}{r-k}G_{k,j}(y)=G_{r,j}(x+y).$$

Noting this lemma and that

$$\sum_{j=0}^{r-1}G_{r,j}(z+N-1)\left\{\frac{(z+N)^{j+1}}{j+1}-\frac{N^{j+1}}{(j+1)^2}\right\}=\int_N^{z+N}\frac{dv}{v}\int_0^v\binom{z+N-1-u}{r-1}du,$$

we rewrite the logarithms of terms in brackets of (7) and have the following asymptotic behavior as $N\to\infty$:

$$\log\left[\prod_{n=1}^{N}\frac{G_{r-1}(n)}{G_{r-1}(z+n)}\prod_{k=0}^{r-1}G_k(N+1)^{\binom{z}{r-k}}\right]=$$

$$= \log G_r(z+1) + \sum_{k=0}^{r} \binom{z}{r-k} \left\{ \binom{N}{k} + \sum_{j=0}^{r-1} \frac{B_{j+1}}{j+1} G_{k,j}(N-1) \right\} \log N -$$

$$- \sum_{j=0}^{r-1} \sum_{k=0}^{r} \binom{z}{r-k} G_{k,j}(N-1) \frac{N^{j+1}}{j+1} - \sum_{j=0}^{r-1} \sum_{k=0}^{r} \binom{z}{r-k} G_{k,j}(N-1) \zeta'(-j) -$$

$$- \left\{ \binom{z+N}{r} + \sum_{j=0}^{r-1} \frac{B_{j+1}}{j+1} G_{r,j}(z+N-1) \right\} \log(z+N) +$$

$$+ \sum_{j=0}^{r-1} G_{r,j}(z+N-1) \frac{(z+N)^{j+1}}{j+1} + \sum_{j=0}^{r-1} G_{r,j}(z+N-1) \zeta'(-j) + O(N^{-1}) =$$

$$= \log G_r(z+1) + \int_0^z \frac{du}{u+N} \left[\binom{z+N}{r} + \sum_{j=1}^{r} \frac{B_{j+1}}{j+1} G_{r,j-1}(z+N-1) - \right.$$

$$\left. - \int_{-1}^{z+N-1} \binom{v}{r-1} dv + \int_u^z \binom{z-1-v}{r-1} dv \right] + O(N^{-1}).$$

As $N \to \infty$, this integral vanishes because of the following lemma, which was already shown in [13]:

Lemma 2.4 (**Ueno-Nishizawa**) *For arbitrary* $z \in \mathbb{C}$, *we have*

$$\binom{z}{r} + \sum_{j=0}^{r-1} \frac{B_{j+1}}{j+1} G_{r,j}(z-1) = \int_{-1}^{z-1} \binom{t}{r-1} dt + \sum_{j=0}^{r-1} \frac{B_{j+1}}{j+1} G_{r,j}(-1).$$

Therefore, we have proved theorem 2.1.

3 Multiplication formula

As an application of Gauss' product representation, we demonstrate the multiplication formula of the multiple gamma function.

Theorem 3.1

$$\prod_{q_1,\ q_2,\cdots q_r=0}^{p-1} G_r\left(\frac{z+q_1+\cdots+q_r}{p}\right) = \frac{e^{\phi_r(z)}}{p^{\psi_r(z)}} G(z) \tag{10}$$

where

$$\phi_r(z) = \sum_{j=0}^{r-1} \left[\sum_{q_1,\cdots,q_r=0}^{p-1} G_{r,j}\left(\frac{z+q_1+\cdots+q_r}{p} - 2\right) - G_{r,j}(z-1) \right] \zeta'(-j)$$

$$\psi_r(z) = \binom{z}{r} + \sum_{j=0}^{r-1} \frac{B_{j+1}}{j+1} G_{r,j}(z-1).$$

Proof. From the infinite product representation (7), it follows that

$$\prod_{q_1,\cdots,q_r=0}^{p-1} G_r\left(\frac{z+q_1+\cdots+q_r}{p}\right) =$$
$$= \lim_{N\to\infty}\left[\frac{\prod_{n=1}^{p(N-1)-1} G_{r-1}(n)}{\prod_{n=1}^{p(N-1)-1} G_{r-1}(z+n-1)} \times \prod_{k=0}^{r-1} G_r(p(N-1))^{\binom{z-1}{r-k}}\right] \times$$
$$\times \lim_{N\to\infty}\left[\prod_{k=0}^{r} \frac{G_r(N)^{\sum_{q_1,\cdots,q_r}\binom{(z+q_1+\cdots+q_r)/p-1}{r-k}}}{G_r(p(N-1))^{\binom{z-1}{r-k}}} \times \frac{p^{\sum_{m=0}^{pN-1}\psi_{r-1}(z+m)}}{e^{\sum_{m=0}^{pN-1}\phi_{r-1}(z+m)}}\right]$$

We substitute the asymptotic expansion (9) to the logarithm of terms in the second brackets.

$$\log\left[\prod_{k=0}^{r} \frac{G_r(N)^{\sum_{q_1,\cdots,q_r}\binom{(z+q_1+\cdots+q_r)/p-1}{r-k}}}{G_r(p(N-1))^{\binom{z-1}{r-k}}} \times \frac{p^{\sum_{m=0}^{pN-1}\psi_{r-1}(z+m)}}{e^{\sum_{m=0}^{pN-1}\phi_{r-1}(z+m)}}\right] =$$
$$= \left\{\sum_{q_1,\cdots,q_r}\binom{(z+q_1+\cdots+q_r)/p-1}{r} + \sum_{j=0}^{r-1}\frac{B_{j+1}}{j+1}G_{r,j}\left(\frac{z+q_1+\cdots+q_r}{p}+N-2\right)\right\}\log N-$$
$$-\left\{\binom{z+p(N-1)-1}{r} + \sum_{j=0}^{r-1}\frac{B_{j+1}}{j+1}G_{r,j}\left((z+p(N-1)-2)\right)\right\}\log\left(N-1-\frac{1}{p}\right)-$$
$$-\sum_{j=0}^{r+1}\left[\sum_{q_1,\cdots,q_r} G_{r,j}\left(\frac{z+q_1+\cdots+q_r}{p}+N-2\right) - G_{r,j}(z+p(N-1)-1)\right]\frac{N^{j+1}}{(j+1)^2}-$$
$$+\left\{\binom{z+p(N-1-1)}{r} + \sum_{j=0}^{r-1}\frac{B_{j+1}}{j+1}G_{r,j}(z+p(N-1)-1) - \sum_{m=0}^{p(N-1)-1}\psi_r(z+m)\right\}\log p-$$
$$-\sum_{j=0}^{r-1}\left[\sum_{q_1,\cdots,q_r=0}^{p-1} G_{r,j}\left(\frac{z+q_1+\cdots+q_r}{p}+N-2\right) - G_{r,j}(z+p(N-1)-1)\right]\zeta'(-j)-$$
$$-\sum_{m=0}^{pN-1}\phi_r(z+m)+o(1).$$

We show that its divergent terms vanish. First, we compute terms including $\log p$.

Proposition 3.2 *If we dene $\psi_0(z)=0$ and*

$$\psi_r(z) := \binom{z}{r} + \sum_{j=0}^{r-1}\frac{B_{j+1}}{j+1}G_{r,j}(z-1),$$

then *$\psi_r(z)$ satises $\psi_0(z)=z$ and*

$$\binom{p(N-1)-1}{r} + \sum_{j=0}^{r-1}\frac{B_{j+1}}{j+1}G_{r,j}(z+p(N-1)-2) - \sum_{m=0}^{p(N-1)-1}\psi_{r-1}(z+m) = \psi_r(z).$$

$\psi_r(z)$ does not depend on N and is uniquely determined as the polynomial satisfying the above recurrence relation.

Proof. This proposition immediately follows from the relation

$$\sum_{l=0}^{L-1}\binom{x+k}{k}=\binom{z+L}{k+1}-\binom{z}{k+1}$$

for $L\in\mathbb{Z}_{\geq 0}$.

Next, we simplify terms including $\zeta'(-j)$ and give a explicit form of $\phi_r(z)$.

Proposition 3.3 *If we dene*

$$\phi_{r,j}(z):=\sum_{q_1,\cdots,q_r=0}^{p-1}G_{r,j}\left(\frac{z+q_1+\cdots+q_r}{p}-2\right)-G_{r,j}(z-1),$$

then $\phi_r(z)=\sum_{j=0}^{r-1}\phi_{r,j}(z)\zeta'(-j)$ *is uniquely determined as a polynomial satisfying the recurrence relation* $\phi_0(z)=0$ *and*

$$\sum_{j=0}^{r-1}\left[\sum_{q_1,\cdots,q_r=0}^{p-1}G_{r,j}\left(\frac{z+q_1+\cdots+q_r}{p}+N-2\right)-G_{r,j}(z+p(N-1)-1)\right]\zeta'(-j)-$$
$$-\sum_{m=0}^{pN-1}\phi_{r-1}(z+m)=\phi_r(z)$$

Proof. It is sufficient to prove

$$\phi_{r,j}(z)=\sum_{q_1,\cdots,q_r}G_{r,j}\left(\frac{z+q_1+\cdots+q_r}{p}+N-2\right)-G_{r,j}(z+p(N-1)-1)-$$
$$-\sum_{m=0}^{p(N-1)-1}\left[\sum_{q_1,\cdots,q_r}G_{r-1,j}\left(\frac{z+m+q_1+\cdots+q_{r-1}}{p}+N-2\right)-G_{r-1,j}(z+m+p(N-1)-1)\right].$$

We can see from the identity

$$\sum_{m=0}^{L}G_{r,j}(z+m)=G_{r+1,j}(z+L)-G_{r+1,j}(z),\quad (L\in\mathbb{Z}_{\geq 0}),$$

and

$$\sum_{m=0}^{p(N-1)-1}\sum_{q_1,\cdots,q_{r-1}=0}^{p-1}G_{r,j}\left(\frac{z+m+q_1+\cdots+q_{r-1}}{p}-2\right)=$$
$$=\sum_{q_1,\cdots,q_{r-1},q=0}^{p-1}\left[G_r\left(\frac{z+q_1+\cdots+q_r}{p}+N-2\right)-G_r\left(\frac{z+q_1+\cdots+q_r}{p}-2\right)\right],$$

The uniqueness of $\phi_r(z)$ follows from its polynomiality.

In order to finish our proof, we verify that the rest of terms vanish as $N\to\infty$. By lemma 2.3, we can see that

$$\sum_{k=0}^{r}\left\{\sum_{q_1,\cdots,q_r=0}^{p-1}\binom{(z+q_1+\cdots+q_r)/p-1}{r-k}\right\}\times$$

$$\times\left[\binom{N}{k}+\sum_{j=0}^{k-1}\frac{B_{j+1}}{j+1}G_{k,j}(N-1)-\sum_{j=0}^{r}G_{k,j}(N-1)\frac{N^2}{(j+1)^2}\right]-$$

$$-\sum_{k=0}^{r}\binom{z-1}{r-k}\left\{\binom{N}{k}+\sum_{j=0}^{r-1}\frac{B_{j+1}}{j+1}G_{k,j}(N-1)-\sum_{j=0}^{k-1}G_{k,j}(N-1)\frac{N^{j+1}}{(j+1)^2}\right\}$$

$$=\sum_{q_1,\cdots,q_r=0}^{p-1}\left[\binom{(z+q_1+\cdots+q_r)/p-1}{r}+\sum_{j=0}^{r-1}\frac{B_{j+1}}{j+1}G_{r,j}\left(\frac{z+q_1+\cdots+q_r}{p}+N-2\right)-\right.$$

$$\left.-\sum_{j=0}^{r-1}G_{r,j}\left(\frac{z+q_1+\cdots+q_r}{p}+N-2\right)\frac{N^2}{(j+1)^2}\right]-$$

$$-\left\{\binom{z+N-1}{r}+\sum_{j=0}^{r-1}\frac{B_{j+1}}{j+1}G_{r,j}(z+N-2)-\sum_{j=0}^{r-1}G_{r,j}(z+N-2)\frac{N^{j+1}}{(j+1)^2}\right\}.$$

From the same argument as proof of theorem 2.1, it follows that the above terms tend to zero as $N\to\infty$. Therefore, we have proved theorem 3.1.

Our result is closely related with Kuribayashi [7]. In order to explain his result, we introduce some functions. $\zeta_r(s,z)$ is defined as a special case of Barnes' zeta function [5, 14], which is introduced as the series

$$\zeta_r(s,z):=\sum_{n_1,\cdots,n_r=0}^{\infty}(z+n_1+\cdots+n_r)^{-s}$$

for $\Re s > r$. This function can be continued analytically to a meromorphic function whose poles are placed at $s=1,\cdots,r$. We call the analytic continuation also $\zeta_r(s,z)$. The gamma function $\Gamma_r(z)$ associated with $\zeta_r(s,z)$ is introduced as

$$\Gamma_r(z):=\exp\left[\frac{\partial}{\partial s}\zeta_r(s,z)\bigg|_{s=0}\right].$$

Kuribayashi exhibit the following multiplication formula:

Theorem 3.4 (**Kuribayashi**) $\Gamma_r(z)$ *satises the following multiplication formula*:

$$\prod_{q_1,\cdots,q_r=0}^{p-1}\Gamma_r\left(\frac{z+q_1+\cdots+q_r}{p}\right)=p^{Q_r(z)}\Gamma_r(z),$$

where

$$Q_r(z)=\frac{(-1)^r}{(r-1)!}\sum_{r=1}^{r}\frac{{}_rS_l}{l}\left\{z^l-(-1)^lB_l\right\}.$$

As a consequence of facts in Vardi [14], a relation between $G_r(z)$ and $\Gamma_r(z)$ is expressed as follows:

$$G_r(z)=R_r(z)\Gamma_r(z)^{(-1)^{r-1}} \quad \text{where} \quad R_r(z):=\exp\left[\sum_{j=0}^{r-1}G_{r,j}(z-1)\zeta'(-j)\right].$$

Thus, we have

$$Q_r(z)=(-1)^r\psi_r(z)=(-1)^r\left[\binom{z}{r}+\sum_{j=0}^{r-1}\frac{B_{j+1}}{j+1}G_{r,j}(z-1)\right]. \tag{11}$$

Our expression is useful in some cases of studies on related functions. For example, noting that $G_{r,0}(z)=\binom{z}{r-1}$, we can check that the relation follows

$$(-1)^rQ_r(r-z)=Q_r(z). \tag{12}$$

from the definition of $\psi_r(z)$ and (11). It plays an important role in the multiplication formula

$$\prod_{q_1,\cdots,q_r=0}^{p-1}S_r\left(\frac{z+q_1+\cdots+q_r}{p}\right)=S_r(z).$$

for Kurokawa's multiple sine function [8, 9, 10, 11] introduced as

$$S_r(z):=\Gamma_r(r-z)\Gamma_r(z)^{(-1)^{r+1}}.$$

In Kuribayashi's original proof, (12) is verified through a rather complicated argument, He applied a relation between $\zeta_r(-m,z)$ $(m\in\mathbb{Z}_{\geq 0})$ and the Bernoulli polynomials $B_l(z)$. However, once (11) is obtained, we can check (12) immediately.

4 Appendix : an elementary proof for (11)

Without facts of zeta functions, we can prove (11) directly as follows: First, we rewrite Kuribayashi's $Q_r(z)$ as

$$(-1)^rQ_r(z)=\frac{1}{(r-1)!}\sum_{l=0}^{r-1}{}_{r-1}S_l\left\{\frac{(-1)^{l+1}B_{l+1}}{l+1}-\frac{(z-1)^{l+1}}{l+1}\right\}. \tag{13}$$

The second term can be written as follows:

$$\frac{1}{(r-1)!}\sum_{l=0}^{r-1}{}_{r-1}S_l\frac{(z-1)^{l+1}}{l+1}=\int_0^z\binom{t-1}{r-1}dt-\int_0^1\binom{t-1}{r-1}dt.$$

From Lemma 2.4 and

$$G_{r,j}(0)=\frac{(-1)^j}{(r-1)!}{}_{r-1}S_j,$$

it follows that

$$\int_0^z\binom{t-1}{r-1}dt-\int_0^1\binom{t-1}{r-1}dt=\binom{z}{r}+\sum_{j=0}^{r-1}\frac{B_{j+1}}{j+1}G_{r,j}(z-1)-\frac{1}{(r-1)!}\sum_{j=0}^{r-1}\frac{B_{j+1}}{j+1}(-1)^j{}_{r-1}S_j.$$

Therefore, we obtain (11) by substituting this to (13).

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