On the nilpotency index of the radical of a group algebra. XII

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J. London Math. Soc. に掲載された同名の論文 III の finite Dickson near fields への拡張。

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On the nilpotency index of the radical of a group algebra. XII [1]

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We gave negative answers in [3] to Tsushima's problem [8, Problems 3, 4, 5]. This example was an affine group over a finite field. In this paper, we shall show that a finite field, which is a basis of this affine group, can be extended to a finite Dickson near field.

Let $t(G)$ be the nilpotency index of the radical $J(KG)$ of a group algebra KG of a finite p-solvable group G over a field K of characteristic $p > 0$ containing a primitive $|G|$-root of 1. Then it is well known by D. A. R. Wallace [9] that

$$p^r \geq t(G) \geq r(p-1)+1$$

where p^r is the order of a Sylow p-subgroup of G.

Let H be a sharply 2-fold transitive group on $\Delta = \{0, 1, \alpha, \beta, \ldots, \gamma\}$ (see [10, p.22]), let $V = H_0$ be a stabilizer of 0 and let U be the set consisting of the identity ε and fixed point-free permutations in H. Then U is a normal and elementary abelian p-subgroup of H with the order p^a (see [6, Lemma 1]). Let σ be a permutation of order p on Δ satisfying conditions

$$\sigma H \sigma^{-1} \subseteq H,\ \sigma^p = 1,\ \sigma(0) = 0,\ \text{and}\ \sigma(1) = 1.$$

Then it is easy to see $\sigma U \sigma^{-1} \subseteq U$ and $\sigma V \sigma^{-1} \subseteq V$. We set $W = <\sigma>$ and $C_V(\sigma) = \{v \in V \mid \sigma v = v\sigma\}$. Assume that there exists a normal subgroup T of WV contained in V such that V is a semi-direct product of T by $C_V(\sigma)$. We set $G = <W,\ T,\ U> = WTU$.

We proved the following results (see [6]).

I. Δ is a near field of characteristic p with respect to the proper sum and product.

II. *σ is an automorphism of* Δ.

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III. *Let p^{a+1} be the order of a Sylow p-subgroup WU of G. Then* $t(G) = (a+1)(p-1)+1$.

We can be really construct a group satisfying the above conditions (see [6, Example 1]). Conversely, the following Theorem 1 shows that our considering group G is just in [6, Example 1].

Theorem 1. *Δ is a finite Dickson near field with the basic finite field $\boldsymbol{F} = \boldsymbol{F}_{q^{pn}}$ where $q = p^r$ for a prime p and $\sigma(x) = x^{q^n}$ for $x \in \Delta$. We consider some permutations on Δ.*

$$u_c : x \to x + c \textit{ for } c \in \Delta, \quad v_c : x \to cx \textit{ for } c \in \Delta^*.$$

Then we obtain

$$U = \{u_c \mid c \in \Delta\},\ V = \{v_c \mid c \in \Delta^*\}, \quad \textit{and} \quad H = < U, V > .$$

and we may set

$$T = \{v_c \in V \mid c \in < a^{\frac{q^n-1}{n}} >\} \textit{ and } G = < W,\ T,\ U > .$$

Proof. We can see from the results I, II and the classification of finite near fields (see [11]) that Δ is a Dickson near field because Δ has an automorphism of order p where p is the characteristic of Δ.

Thus Δ has the basic finite field $\boldsymbol{F} = \boldsymbol{F}_{q^s}$ where $q = p^r$ for a prime p and σ is also an automorphism with order p of a finite field $\boldsymbol{F}$. Hence $s = pn$ for an integer n, $\sigma(x) = x^{q^n}$ for $x \in \boldsymbol{F}$ and $\boldsymbol{F}_{q^n}$ is the field fixed by σ. We may assume that (q, n) and (q^p, n) are Dickson pairs because p is not divisor of $q-1$ and changes of values of r and n with the constant rn. Thus Δ is a Dickson near field defined by the automorphism $\tau : x \to x^{q^p}$ of $\boldsymbol{F}$. Let ω be a generator of the multiplicative group $\boldsymbol{F}^*$ and we set $a = \omega^n$, $b = \omega$ in $\boldsymbol{F}^*$. Then the multiplicative group Δ^* of Δ has the structure

$$\Delta^* = < a,\ b \mid a^m = 1,\ b^n = a^t,\ bab^{-1} = a^{q^p} >$$

where $m = \frac{q^{pn}-1}{n}$, $t = \frac{m}{q^p-1}$. Here we use the usual symbol as the product in Δ for simplicity. Do not confuse with the product in $\boldsymbol{F}$. We consider some permutations on Δ.

$$u_c : x \to x + c \text{ for } c \in \Delta, \quad v_c : x \to cx \text{ for } c \in \Delta^*.$$

Then we have some relations

$$u_c u_d = u_{d+c},\ v_c v_d = v_{cd},\ v_c u_d v_c^{-1} = u_{cd},\ \sigma u_c \sigma^{-1} = u_{\sigma(c)},\ \sigma v_c \sigma^{-1} = v_{\sigma(c)}$$

on u_c, v_c, σ. We can see from [6, proof of Theorem 3] that

$$U = \{u_c \mid c \in \Delta\},\ V = \{v_c \mid c \in \Delta^*\},\ \text{ and } H = UV.$$

We set

$$T = \{v_c \in V \mid c \in < a^{\frac{q^n-1}{n}} >\}.$$

It is easy to see that $H = UV$ is sharply 2-fold transitive on Δ, T is normal in WV and the order of T is $\frac{q^{pn}-1}{q^n-1}$ because products of a and x in Δ are the same in $\boldsymbol{F}$. On the other hand, the set $C_V(\sigma)$ is equal to $\boldsymbol{F}^*_{q^n}$ as a set and the order of $C_V(\sigma)$ is $q^n - 1$. Since $\frac{q^{pn}-1}{q^n-1}$ and $q^n - 1$ are relatively prime, we have $V = C_V(\sigma)T,\ C_V(\sigma) \cap T = \{\varepsilon\}$.

Let $X =< x >$ be the cyclic group of order $|T|$ and let τ be an automorphism of X defined by $\tau(x^k) = x^{kq^n}$. Then $\sigma \to \tau$ is a homomorphism of W into $\mathrm{Aut}(X)$. We can regard this as a homomorphism of G. Let N be a semi-direct product of X by G with respect to this homomorphism. In the remainder part of this paper, ζ will represent a primitive $|T|$-th root of 1 in K.

Lemma 1. *$t(N) = (rpn + 1)(p - 1) + 1$, where $p^{rpn} = |U|$.*

Proof. First of all, we shall observe that $f_k = \sum_{i=0}^{\ell-1} \zeta^{ik} x^i (0 \le k < |T|)$ are all orthogonal primitive idempotents of KX. We can see that

$$I_N(f_k) = \{x \in N | x f_k x^{-1} = f_k\} = \begin{cases} TUV & \text{if} \quad k \neq 0, \\ N & \text{if} \quad k = 0. \end{cases}$$

Indeed, if $I_N(f_k)$ properly contains TUV, then $I_N(f_k)$ must contain σ and so $f_k^\sigma = f_k$. Since the coefficient of x in $f_k^\sigma - f_k$ is $\zeta^{kq^{n(p-1)}} - \zeta^k$, it follows that $k \equiv q^n k \bmod |T|)$, which shows that $k = 0$ because $|T| = \frac{q^{np}-1}{q^n-1}$ by Theorem 1 and $(|T|, q^n - 1) = 1$. From Morita's theorem [2, Theorem 2], the formula (1) asserts that KN is isomorphic to a direct sum of a group algebra KG and full matrix algebras over twisted group algebras $K^\alpha UV$ with factor sets α. By [7, Theorem 1.6] , $J(K^\alpha UV) = J(KU)K^\alpha UV$ and hence the nilpotency index of $J(K^\alpha UV)$ is equal to $t(U) = rpn(p-1)+1$, where $p^{rpn} = |U|$. This implies the desired conclusion by result III because $p^{rpn+1} = |WU|$ is the order of a p-Sylow subgroup of N.

Let c be a generator of T and let Y be a direct product of two cyclic groups $< x >$ and $< y >$ which have the same order ℓ. Let ϕ and ψ are automorphisms of Y defined by

$$\phi(x) = x,\ \phi(y) = xy;\ \psi(x) = x,\ \psi(y) = y^q.$$

Then we have $\phi^\ell = 1, \psi^p = 1$ and $\phi\psi = \psi\phi^q$. Hence $c \to \phi, \sigma \to \psi^{-1}$ defines a homomorphism of TW into $\mathrm{Aut}(Y)$. We can regard this as a homomorphism of G. Let M be a semi-direct product of Y by G with respect to this homomorphism.

The next result gives negative answers to Tsushima's problem [8, Problems 3, 4, 5]. This counter example is a group that yields from an affine group over a finite Dickson near field.

Theorem 2. *A p'-element x is contained in the center of M and*

$$t(M) - t(M/ < x >) \geq rn(p^2 - 1).$$

Proof. From the definition of M, we have the following relations:

$$x^c = x, y^c = xy, x^\sigma = x, y^{\sigma^{-1}} = y^q.$$

Thus the first assertion is trivial. Let f be a primitive idempotent of KY defined by

$$f = (1 + \zeta x + \zeta^2 x^2 + \cdots + \zeta^{\ell-1} x^{\ell-1})(1 + y + y^2 + \cdots + y^{\ell-1}).$$

Then $f^\sigma = f$ and $f^c = \zeta f \neq f$. Thus $I_M(f) = PY$ where $P = WU$. Hence, Morita's theorem [2, Theorem 2] implies that $t(M) \geq t(P)$. Using Jennings' formula [1], since $< P', P^p > = \{u_{b q^n - b} | b \in F\}$, we obtain that $t(P) \geq (2rnp - rn)(p-1) + 1$. With these observations, the result follows from Lemma 1.

Remark. If a group G has a 3-Sylow subgroup $M(3) = < a, b, c \mid a^3 = b^3 = c^3 = 1,\ aba^{-1} = b,\ aca^{-1} = bc,\ bc = cb >$, then we can see from [4, Theorem] that $t(G) = t(G/M) = 9$ for any $3'$-normal subgroup M of G. Hence we can not construct a counter example for Tsushima's problem [8, Problems 3, 4, 5].

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