# On the nilpotency index of the radical of a group algebra

K. Motose

岡山大学での Symposium における Program, Abstract と口頭発表 (Oct. 13, 2001) の OHP 原稿。

June 2003 本瀬香

# 第３５回 環論および表現論シンポジウム（第二報）

日本学術振興会科学研究費補助金 基盤研究 (B)(1)(研究代表者：西田 憲司 (信州大学理学部)) による上記シンポジュウムのプログラムが出来ましたのでお知らせいたします。また、懇親会も下記のように計画しましたのであわせてご案内申し上げます。

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------------ 記 ------------

日程： 2002 年 １０ 月 １２ 日（土）～１４ 日（月）
場所： 岡山大学環境理工学部棟１０４講義室　〒700-8530 岡山市津島中 3-1-1
電話 086-252-1111 (http://www.okayama-u.ac.jp/Location/tsushima_j.html)
（会場責任者 池畑秀一（岡山大学環境理工学部））

（JR 岡山駅前から岡電バス「岡山大学・妙善寺」行に乗車、「岡大東門」で下車、または
JR 岡山駅西口から岡電バス「岡山理科大学」行に乗車、「岡大東門」で下車 ）

懇親会： １０月１３日（日）岡山大学生協ピーチユニオン３階で１８時３０分から

懇親会（会費 4,000 円）に参加ご希望の方は、９月２７日（金）までに、下記宛にお申込下さい。多数の方のご参加をお待ちしています。

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# 第 35 回 環論および表現論シンポジウム

## プログラム

10月12日（土）

**9:00–9:30**　柳井　忠（新居浜工業高等専門学校）

ホップ加群の双対性とその応用

(Hopf module duality and its application)

**9:40–10:10**　和久井　道久（大阪大学大学院理学研究科）

9 次元以下の半単純でないホップ代数の表現環について

(On represenation rings of non-semisimple Hopf algebras of low dimension)

**10:20–10:50**　雪本　義人

ネーター環を特徴づける一つの条件とその双対

(A characterization of Noetherian rings and its dual)

**11:00–12:00**　Kenneth R. Goodearl (University of California at Santa Barbara)

Quantized coordinate rings and related noetherian algebras I

**13:20–13:50**　増岡　彰（筑波大学数学系）

接合積からのホップ代数入門

(An introduction to Hopf algebras via crossed products)

**14:00–14:30**　土井　幸雄（岡山大学教育学部）

群環的代数

(Group-like algebras)

**14:40–15:10**　星野　光男（筑波大学数学系）

加藤　義明（筑波大学大学院数学研究科）

An elementary construction of tilting complexes

**15:30–16:00**　藤田　尚昌（筑波大学数学系）

Neat idempotents and tiled orders having large global dimension

**16:10–16:40**　長瀬　潤（大阪市立大学大学院理学研究科）

Algebra homomorphisms and Hochschild cohomology

**16:50–17:20**　速水　孝夫（東京理科大学大学院理学研究科）

真田　克典（東京理科大学理学部）

Cohomology rings of the generalized quaternion group

**17:30–18:00**　奥山　京（鳥羽商船高等専門学校）

無限アーベル群における mixed 群について

(Mixed groups in Abelian group theory)

１０月１３日（日）

**9:00–9:30**　小林　滋（鳴門教育大学）

丸林　英俊（鳴門教育大学）

Nicolae Popescu (Institute of Mathematics of the Romanian Academy)

Guanging Xie（鳴門教育大学）

Total valuation rings of Ore extensions

**9:40–10:10**　Miguel A. Ferrero (Universidade Federal do Rio Grande do Sul)

The unitary strongly prime rings and related radicals

**10:20–10:50**　Nicolae Popescu (Institute of Mathematics of the Romanian Academy)

Transitive Galois action on plane compacts

**11:00-12:00**　Kenneth R. Goodearl (University of California at Santa Barbara)

Quantized coordinate rings and related noetherian algebras II

**13:20-13:50**　荒谷　督司（岡山大学大学院自然科学研究科）

非可換環上の Cohen-Macaulay 次元

(Cohen-Macaulay dimensions over non-commutative rings)

**14:00-14:30**　高橋　亮（岡山大学大学院自然科学研究科）

吉野　雄二（岡山大学理学部）

Looking at homological dimensions through Frobenius map

**14:40–15:10** 吉川　昌慶（信州大学大学院工学系研究科）

ハミング・アソシエーションスキームのモジュラー隣接代数

(Modular adjacency algebras of the Hamming association schemes)

**15:30–16:00** 宮本　雅彦（筑波大学数学系）

Symmetric algebra and modular invariance of VOA

**16:10–16:40** 水川　裕司（北海道大学大学院理学研究科）
山田　裕史（岡山大学理学部）

Kac-Moody Lie 環の基本表現と Schur 函数

(Plethysm of Schur Functions and The Basic Representation of $A_2^{(2)}$)

**16:50–17:20** Ziqun Lu (魯　自群)（北京大学及び千葉大学理学部）

Monomial modules and endo-monomial modules

**17:30–18:00** 本瀬　香（弘前大学理工学部）

On the nilpotency index of the radical of a group algebra

１０月１４日（月）

**9:00–9:30** 倉富　要輔（山口大学理工学研究科）

Direct sums of lifting modules

**9:40–10:10** 千葉　克夫（新居浜工業高等専門学校）

自由体とその付値

(Free fields in complete skew fields and their valuations)

**10:20–10:50** 中島　晴久 (城西大学理学部)

Equidimensional actions of algebraic tori on graded normal domains

**11:10–12:10** Kenneth R. Goodearl (University of California at Santa Barbara)

Quantized coordinate rings and related noetherian algebras III

**12:20–12:50** 関口　勝右 (国士舘大学工学部)

Extensions of some 2-groups which preserve the irreducibilities of induced characters

On the nilpotency index of the radical of a group algebra

Kaoru Motose

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Let $t(G)$ be the nilpotency index of the radical of a group algebra of a finite $p$-solvable group $G$ over a field of characteristic $p$. Then it is well known that

$$p^s \geq t(G) \geq s(p-1)+1$$

where $p^s$ is the order of a $p$-Sylow subgroup of $G$.

H. Fukushima [1] characterized a group $G$ of $p$-length 2 satisfying $t(G) = s(p-1)+1$ under a condition such that

$$\text{the } p'\text{-part } V = O_{p',p}(G)/O_p(G) \text{ is } \textit{abelian}.$$

In this talk, we shall show that the next group $G$ of $p$-length 2 with the *non abelian* $p'$-part $V$ satisfies $t(G) = s(p-1)+1$ where $s = rpn+1$.

This new example will contribute to our research.

Let $(q,n)$ be a Dickson pair where $p$ is a prime and $q = p^r$. Then $(q^p, n)$ is also a Dickson pair. Let $\boldsymbol{D} = \boldsymbol{D}_{q^{pn}}$ be a finite Dickson near field defined by a finite field of order $q^{pn}$ and its automorphism $x \to x^{q^p}$.

The following affine group $G$ over $\boldsymbol{D}$ is our object (see [2]).

$$G = \{x \to a x^{q^{nk}} + b \mid a \in \boldsymbol{D}^*,\ b \in \boldsymbol{D},\ k = 0, \ldots, p-1\}$$

## References

1. H. Fukushima, On groups $G$ of $p$-length 2 whose nilpotency indices of $J(KG)$ are $a(p-1)+1$, Hokkaido Math. J. 20(1991), 523-530.

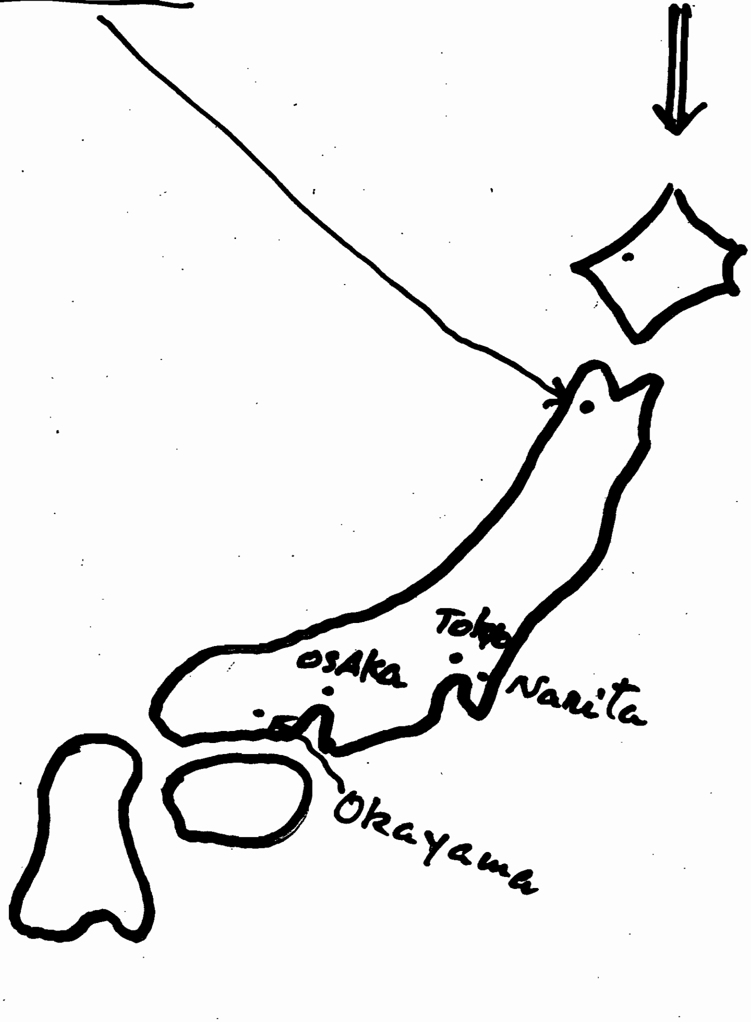
2. K. Motose, On the nilpotency index of the radical of a group algebra. III, J. London Math. Soc. (2), 25(1982), 39-42.

# On the nilpotency index of the radical of a group algebra


Kaoru Motose

Hirosaki Univ JAPAN


⇓ Topological Map of JAPAN



# $\boxed{1}$ Notations and known results

1—1 · $\underline{G}$ : f. gp, · p : a fixed prime
$p \mid |G|$

· $\underline{K}$ : field of char p
$\mathbb{F}_q$ : finite field of order q
· $\underline{KG}$ : group alg. of G
over K
· $\underline{J(KG)}$ : radical of KG

· $\underline{t(G)}$ : nilpotency index
of J(KG)
i.e. ~~the~~ least smallest number k
s.t. $J(KG)^k = 0$
· $\underline{S}$ : p-Sylow subgp of G

· $\underline{p^a} = |S|$

⊙ $G \rhd H$, $p \nmid |G:H| \Rightarrow$

$J(KG) = J(KH)KG$, $t(G) = t(H)$

⊙ $G$ : Frob gp, $N$ : Kernel, $H$ : complement

$\Rightarrow J(KG) = J(KH)\hat{N}$ for $p \mid |H|$

Where $\hat{N} = \sum_{x \in N} x$

⊙ $G$ : p-solv $\Longrightarrow \underset{|S|}{\underset{\shortparallel}{p^a}} \geq t(G) \geq a(p-1)+1$

⊙ $G$ : p-solv with p-length 1

$\Rightarrow$

$t(G) = a(p-1)+1 \Leftrightarrow S$ : elementary abelian

✡ $t(G) = a(p-1)+1$ and $S$ : not abelian

We can find such a group

construct

$p$ : a prime

$\mathbb{F} = \mathbb{F}_{p^{rp}}$ : finite field of order $p^{rp}$

$\sigma : x \to x^{p^r}$, $u_a : x \to x + a$ $(a \in \mathbb{F})$

$v_a : x \to ax$ $(a \in F^*)$

permutations on $\mathbb{F}$

$W := \langle \sigma \rangle$, $V := \{ v_a \mid a \in A \}$

$U := \{ u_a \mid a \in \mathbb{F} \}$, $A := \langle w^{p^r - 1} \rangle$

where $\mathbb{F}^* = \langle w \rangle$

$$G = \langle W, V, U \rangle$$
$$= \{ x \to a x^{p^{rk}} + b \mid a \in A,\ b \in \mathbb{F},\ k = 0, 1, \cdots, p-1 \}$$

permutation gp on $\mathbb{F}$

affine gp over $\mathbb{F}$

We can prove 4

$|WU| = p^{rp+1}$

$t(G) = (rp+1)(p-1)+1$

<u>Points</u> in PROOF

$(p^r - 1, \frac{p^{rp}-1}{p^r-1}) = 1$

◎ <u>WV: Frob</u>

$\bar{H}^* = \bar{H}^*_{p^r} \cdot A$

$\cup \quad \cup \quad \cup$

$c = b a$

◎ $\sigma$, $\hat{V} u_c \hat{V} = \hat{V} u_b \hat{V}$ $\rightarrow (J(KW)\hat{V}KG)^p = 0$

HENCE

<u>are commutative</u>

∘ $J(KG) = J(KW)\hat{V}KG + J(KU)KG$

$t(G) = (rp+1)(p-1)+1$

$S = WU$: non commutative

- Löwy Series

$P$ : projective Indecom. module of $KG$

$\frac{JP}{J^2P}$ = direct Sum of irreducible modules (c. r. (s.s.) module)

$\frac{J^2P}{J^3P}$

⋮

$\frac{J^{s-1}P}{J^sP}$ $\quad s = t(G)$

**Problem**

Which irreducible modules are appeared?

- For the above gp. Löwy Series are determined.
- Thease are Computed by only Culculation of integers

The number of irr module ≤ 10

Table is completed by Computer (1989)

For Example

5'



$F = GF(2^8)$, $h_0 = h = 17$ and $M$, $P =$ 0 1 2 3 4 5 6 7 8 ≤ 10

P0

0/1=1M0
1/2=1M0 1M1 1M2 1M4 1M8
2/3=4M0 2M1 2M2 2M3 2M4 2M5 2M6 2M7 2M8
3/4=4M0 5M1 5M2 5M3 5M4 5M5 5M6 5M7 5M8
4/5=6M0 7M1 7M2 8M3 7M4 8M5 8M6 8M7 7M8
5/6=6M0 7M1 7M2 8M3 7M4 8M5 8M6 8M7 7M8
6/7=4M0 5M1 5M2 5M3 5M4 5M5 5M6 5M7 5M8
7/8=4M0 2M1 2M2 2M3 2M4 2M5 2M6 2M7 2M8
8/9=1M0 1M1 1M2 1M4 1M8
9/10=1M0

P1

0/1=1M1
1/2=1M0 1M1 1M2 2M3 1M5 1M7 1M8
2/3=2M0 4M1 2M2 2M3 3M4 3M5 4M6 3M7 2M8
3/4=5M0 4M1 6M2 6M3 5M4 5M5 4M6 5M7 6M8
4/5=7M0 6M1 6M2 5M3 7M4 6M5 7M6 6M7 6M8
5/6=7M0 5M1 6M2 5M3 7M4 6M5 7M6 6M7 6M8
6/7=5M0 4M1 6M2 6M3 5M4 5M5 4M6 5M7 6M8
7/8=2M0 4M1 2M2 2M3 3M4 3M5 4M6 3M7 2M8
8/9=1M0 1M1 1M2 2M3 1M5 1M7 1M8
9/10=1M1

P2

0/1=1M2
1/2=1M0 1M1 1M2 1M3 1M4 2M6 1M7
2/3=2M0 2M1 4M2 3M3 2M4 4M5 2M6 3M7 3M8
3/4=5M0 6M1 4M2 5M3 6M4 4M5 6M6 5M7 5M8
4/5=7M0 6M1 6M2 6M3 6M4 7M5 5M6 6M7 7M8
5/6=7M0 6M1 5M2 6M3 6M4 7M5 5M6 6M7 7M8
6/7=5M0 6M1 4M2 5M3 6M4 4M5 6M6 5M7 5M8
7/8=2M0 2M1 4M2 3M3 2M4 4M5 2M6 3M7 3M8
8/9=1M0 1M1 1M2 1M3 1M4 2M6 1M7
9/10=1M2

P3

0/1=1M3
1/2=2M1 1M2 1M4 2M5 1M6 1M7
2/3=2M0 2M1 3M2 6M3 3M4 2M5 3M6 3M7 4M8
3/4=5M0 6M1 5M2 3M3 5M4 8M5 7M6 7M7 4M8
4/5=8M0 5M1 6M2 8M3 6M4 3M5 4M6 4M7 7M8
5/6=8M0 5M1 6M2 3M3 6M4 5M5 4M6 4M7 7M8
6/7=5M0 6M1 5M2 3M3 5M4 6M5 7M6 7M7 4M8
7/8=2M0 2M1 3M2 6M3 3M4 2M5 3M6 3M7 4M8
8/9=2M1 1M2 1M4 2M5 1M6 1M7
9/10=1M3

P4

0/1=1M4
1/2=1M0 1M2 1M3 1M4 2M5 1M6 1M8
2/3=2M0 3M1 2M2 3M3 4M4 2M5 3M6 4M7 2M8
3/4=5M0 5M1 6M2 5M3 4M4 6M5 5M6 4M7 6M8
4/5=7M0 7M1 6M2 6M3 6M4 5M5 6M6 7M7 6M8
5/6=7M0 7M1 6M2 6M3 5M4 5M5 6M6 7M7 6M8
6/7=5M0 5M1 6M2 5M3 4M4 6M5 5M6 4M7 6M8
7/8=2M0 3M1 2M2 3M3 4M4 2M5 3M6 4M7 2M8
8/9=1M0 1M2 1M3 1M4 2M5 1M6 1M8
9/10=1M4

P5

0/1=1M5
1/2=1M1 2M3 2M4 1M6 1M7 1M8
2/3=2M0 3M1 4M2 2M3 2M4 6M5 3M6 3M7 3M8
3/4=5M0 5M1 4M2 8M3 6M4 3M5 7M6 7M7 5M8
4/5=8M0 6M1 7M2 3M3 5M4 8M5 4M6 4M7 6M8
5/6=8M0 6M1 7M2 5M3 5M4 3M5 4M6 4M7 6M8
6/7=5M0 5M1 4M2 6M3 6M4 3M5 7M6 7M7 5M8
7/8=2M0 3M1 4M2 2M3 2M4 6M5 3M6 3M7 3M8
8/9=1M1 2M3 2M4 1M6 1M7 1M8
9/10=1M5

P6

0/1=1M6
1/2=2M2 1M3 1M4 1M5 2M7 1M8
2/3=2M0 4M1 2M2 3M3 3M4 3M5 6M6 2M7 3M8
3/4=5M0 4M1 6M2 7M3 5M4 7M5 3M6 8M7 5M8
4/5=8M0 7M1 5M2 4M3 6M4 4M5 8M6 3M7 6M8
5/6=8M0 7M1 5M2 4M3 6M4 4M5 3M6 5M7 6M8
6/7=5M0 4M1 6M2 7M3 5M4 7M5 3M6 6M7 5M8
7/8=2M0 4M1 2M2 3M3 3M4 3M5 6M6 2M7 3M8
8/9=2M2 1M3 1M4 1M5 2M7 1M8
9/10=1M6

P7

0/1=1M7
1/2=1M1 1M2 1M3 1M5 2M6 2M8
2/3=2M0 3M1 3M2 3M3 4M4 3M5 2M6 6M7 2M8
3/4=5M0 5M1 5M2 7M3 4M4 7M5 8M6 3M7 6M8
4/5=8M0 6M1 6M2 4M3 7M4 4M5 3M6 8M7 5M8
5/6=8M0 6M1 6M2 4M3 7M4 4M5 5M6 3M7 5M8
6/7=5M0 5M1 5M2 7M3 4M4 7M5 6M6 3M7 6M8
7/8=2M0 3M1 3M2 3M3 4M4 3M5 2M6 6M7 2M8
8/9=1M1 1M2 1M3 1M5 2M6 2M8
9/10=1M7

P8

0/1=1M8
1/2=1M0 1M1 1M4 1M5 1M6 2M7 1M8
2/3=2M0 2M1 3M2 4M3 2M4 3M5 3M6 2M7 4M8
3/4=5M0 6M1 5M2 4M3 6M4 5M5 5M6 6M7 4M8
4/5=7M0 6M1 7M2 7M3 6M4 6M5 6M6 5M7 6M8
5/6=7M0 6M1 7M2 7M3 6M4 6M5 6M6 5M7 5M8
6/7=5M0 6M1 5M2 4M3 6M4 5M5 5M6 6M7 4M8
7/8=2M0 2M1 3M2 4M3 2M4 3M5 3M6 2M7 4M8
8/9=1M0 1M1 1M4 1M5 1M6 2M7 1M8
9/10=1M8

# 2 finite Dickson Near fields

## 2-1 Dickson pair

**Def** $(q, n)$ : Dickson pair

① $q = p^e$ : power of a prime $p$

② prime div of $n$ is a div of $q-1$

③ $q \equiv 3 \mod 4 \rightarrow n \not\equiv 0\ (4)$

◎ $\cancel{k}^{\ell} \mod n \rightarrow \dfrac{q^{\ell}-1}{q-1} \mod n$ is a permutasión of $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$

◎ $q^n \equiv 1 \mod n(q-1)$

◎ $n$ is the order of $q$ mod $n$

## 2-2 Dickson Near fields

$0^* a, b \in \mathbb{F}_{q^n} = \mathbb{F}$ : finite fields

$\rho = \rho_1 : x \to x^q$

$a = \omega^s$, $\dfrac{q^{s'}-1}{q-1} \equiv s \mod n$

$\mathbb{F}^* = \langle \omega \rangle$ $\quad s' \mod n$ : Unique

$\rho_a = \rho^{s'}$

$a \circ b = a\rho_a(b) \quad 0 \circ b = 0$

$\mathbb{D}_{q^n} = (\mathbb{F}, +, \circ)$ : near field

**2-3** $a = \omega^n$, $b = \omega$ where $\mathbb{F}^* = \langle \omega \rangle$

$\mathbb{D}^*_{q^n} = \langle a, b \mid a^m = 1, b^n = a^t, bab^{-1} = a^q \rangle$

where $m = \dfrac{q^n-1}{n}$, $t = \dfrac{m}{q-1}$

2-4 automorphism of $\mathbb{D}_{q^n}$ 

- $\nu : x \to x^p$ auto of $\mathbb{F}_{q^n}/\mathbb{F}_p$

$(q, n) \neq (3, 2)$

$\sigma$ : auto of $\mathbb{D}_{q^n}$ $\Leftrightarrow$ $\sigma = \nu^s$, $s \equiv 0 \mod h$

$h = |P|_n =$ the order of $P$ mod $n$

- $\mathrm{Aut}(\mathbb{D}_{3^2}) \simeq S_3$

## $\boxed{3}$ Small Results

- Now, we consider an affine gp over finite Dickson near fields instead of finite fields
- Our last purpose is to determin Löwy Series
- We now determine nilpotency in dex (Löwy Length) of such a group

$(p, n)^{*(3,2)}$ Dickson pair

( $p$: odd prime )

$\mathbb{F} = \mathbb{F}_{p^{pn}}$

$(p^p, n)$ also Dickson pair

$\mathbb{D} = \mathbb{D}_{p^{pn}}$ by auto $\tau : x \to x^{p^p}$ of $\mathbb{F}$

$\sigma : x \to x^{pn}$

$\sigma^p = 1$, $U_c : x \to x + c$, $V_c : x \to c \circ x$

permutations on $\mathbb{D}$

$$\boxed{\begin{array}{l} \sigma U_c \sigma^{-1} = U_{\sigma(c)}, \quad \sigma V_c \sigma^{-1} = V_{\sigma(c)} \\ V_c V_d = V_{c \circ d}, \quad U_c U_d = U_{d+c} \\ V_c U_d V_c^{-1} = U_{c \circ d} \end{array}}$$

$\mathbb{D}_{p^{pn}} = \langle a, b \mid a^m = 1,\ b^n = a^t,$

$bab^{-1} = a^{p^p} \rangle \quad m = \frac{p^{pn} - 1}{n},\ t = \frac{m}{p^p - 1}$

$a \circ s = as$ for $s \in \mathbb{D}$

$$U = \{ u_\alpha \mid \alpha \in \mathbb{F} \}$$

$$G = \langle \sigma, v_a, v_b, U \rangle$$

$\triangledown$ ) index $n$ $\quad p \nmid n$

$$H = \langle \sigma, v_a, U \rangle$$

$\triangledown$ ) index $p^n - 1$ $\quad p \nmid p^n - 1$

$$M = \langle \sigma, v_a^{p^n-1}, U \rangle \simeq \cancel{\text{gp in } \boxed{2}}$$

Affine gp over $\mathbb{F}_{p^{np}}$ defined in $\boxed{1}$

$$t(G) = t(H) = t(M)$$
$$= (pn+1)(p-1)+1$$

This is the first Examples of gp with p-length 2 s.t. $\underbrace{p'\text{-part}}_{\langle v_a, v_b \rangle}$ of G is non-commutative and $t(G)$ is smallest (with respect to $|S| = p^a$)

$(q, n) = (3, 2)$ $\quad \mathrm{Aut}(\mathbb{D}_{3^2}) \simeq S_3$

$\mathbb{F}_{3^2} = \mathbb{F}_3[x]/(x^2+1) = \{ s+ti \mid i^2 = -1,\ s, t \in \mathbb{F}_3 \}$

$\mathbb{D}_{3^2}$ defined by auto $x \to x^3$ of $\mathbb{F}_3$

$\sigma(s+ti) = s+t+ti$

$\sigma$ : auto of $\mathbb{D}_{3^2}$ (of order 3)

$G = \langle \sigma, V, U \rangle \cong SL(2,3) \ltimes \mathbb{F}_3^{(2)}$

$G$ の Sylow 3-subgp $= \langle \sigma, U \rangle$ の
位数 $= \underline{3^3}$ $\quad |\langle \sigma, U \rangle| = 3^3$

$t(G) = 9 > 7 = 3(3-1)+1$

$\uparrow$

J. Algebra 90(1984), 251-258