

On the nilpotency index of the radical of a group
algebra. XI

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Math. J. Okayama Univ., 44(2002), 51-56

いままで, Conference, Symposium 等で発表した内容を詳しく記述した
論文。

June 2003 本瀬香

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Reprinted from MATHEMATICAL JOURNAL OF OKAYAMA UNIVERSITY
Vol. 44, 2002

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Let $t(G)$ be the nilpotency index of the radical $J(KG)$ of a group algebra KG of a finite p -solvable group G over a field K of characteristic $p > 0$. Then it is well known by D. A. R. Wallace [7] that

$$p^e \geq t(G) \geq e(p - 1) + 1,$$

where p^e is the order of a Sylow p -subgroup of G .

H. Fukushima [1] characterized a group G of p -length 2 satisfying $t(G) = e(p - 1) + 1$, see also [4]. Unfortunately, his characterization holds under a condition such that the p' -part $V = O_{p',p}(G)/O_p(G)$ of G is abelian.

In this paper, using Dickson near fields, we shall give an explicit example (see Example 1) such that a group G of p -length 2 has the non abelian p' -part V and satisfies $t(G) = e(p - 1) + 1$. This example will be new and have a contributions in our research. Example 2 is also very interesting because quite different objects (see [3] and [5]) are unified on the ground of Dickson near fields.

Let H be a sharply 2-fold transitive group on $\Delta = \{0, 1, \alpha, \beta, \dots, \gamma\}$ (see [8, p. 22]). Let $V = H_0$ be a stabilizer of 0, and let U be the set consisting of the identity ε and fixed point-free permutations in H . Then U is an elementary abelian p -subgroup of H with the order p^s (see Lemma 1). Let σ be a permutation of order p on Δ satisfying conditions

$$\sigma H \sigma^{-1} \subseteq H, \quad \sigma^p = 1, \quad \sigma(0) = 0 \quad \text{and} \quad \sigma(1) = 1.$$

Then it is easy to see $\sigma U \sigma^{-1} \subseteq U$ and $\sigma V \sigma^{-1} \subseteq V$. We set $W = \langle \sigma \rangle$ and $C_V(\sigma) = \{v \in V \mid \sigma v = v \sigma\}$. Assume that there exists a normal subgroup T of WV contained in V such that V is a semi-direct product of T by $C_V(\sigma)$. We set $G = \langle W, T, U \rangle$.

Now, we present the well known results Lemmas 1 and 2 for completeness of this paper.

Lemma 1. U is a normal and elementary abelian p -subgroup of H .

Proof. First we shall prove, for $k \in \Delta^* = \Delta \setminus \{0\}$, there exists only one $u_k \in U$ with $u_k(0) = k$, equivalently, the following map ν from U to Δ is bijective:

$$\nu: u \rightarrow u(0).$$

This paper was financially supported by the Grant-in-Aid for Scientific Research from Japan Society for the Promotion of Science (Subject No. 1164003).

For $\tau \in U \setminus \{\varepsilon\}$, there exists $\rho \in H_0$ with $\rho(\tau(0)) = k$ since $\tau(0) \neq 0$ and H_0 is transitive on Δ^* . We set $u_k = \rho\tau\rho^{-1}$. Then $u_k \in U$ and $u_k(0) = k$. Thus ν is surjective. It follows from definition of H and U that

$$U = H \setminus \bigcup_{a \in \Delta} (H_a \setminus \{\varepsilon\}), \quad (H_a \setminus \{\varepsilon\}) \cap (H_b \setminus \{\varepsilon\}) = \emptyset \text{ for } a \neq b.$$

Using $|H| = |H_a||a^H| = |H_a||\Delta|$, where a^H is an orbit of a , we can see $|U| = |\Delta|$. Hence ν is injective.

Assume $\eta\tau$ has a fixed point ℓ for $\eta, \tau \in U$. Then we may assume $\ell = 0$ since H is transitive on Δ and $\rho U \rho^{-1} = U$ for $\rho \in H$. Thus $\tau = \eta^{-1}$ follows from $\eta^{-1} \in U$, $\tau(0) = \eta^{-1}(0)$ and the above observation. This means $\eta\tau \in U$. Hence U is a normal subgroup of H because $\rho U \rho^{-1} = U$ for all $\rho \in H$.

Now, we shall show U is elementary abelian. Let p be a prime factor of $|U|$ and let τ be an element of order p in the center of a Sylow p -subgroup of U . We set $\eta \in U \setminus \{\varepsilon\}$. Then there exists $\rho \in H_0$ with $\rho(\tau(0)) = \eta(0)$. Thus $\rho\tau\rho^{-1} = \eta$ follows from $\rho\tau\rho^{-1} \in U$ and $\rho\tau\rho^{-1}(0) = \eta(0)$. Thus the order of every element in U is p or 1 and so η is in the center of a p -group U . Thus U is elementary abelian. \square

The next shows Δ is a near field of characteristic p .

Lemma 2. Δ is a near field of characteristic p and σ is an automorphism of Δ .

Proof. First, we shall prove that Δ is a near field. We can set a structure of a near field in a set Δ by the following method. It follows from Lemma 1 that there exists only one $u_a \in U$ with $u_a(0) = a$ for $a \in \Delta$. It is easy to see that for $a \in \Delta^* = \Delta \setminus \{0\}$, there exists only one $v_a \in V = H_0$ with $v_a(1) = a$. It is clear from definition that $u_0 = v_1 = \varepsilon$.

We define the sum and the product of elements a, b in Δ by using the above v_a and u_b :

$$a + b := u_b(a), \quad ab := v_a(b) \text{ for } a \neq 0 \text{ and } 0b := 0.$$

First we shall prove the next equations:

$$u_a u_b = u_{b+a}, \quad v_a v_b = v_{ab} \quad \text{and} \quad v_a u_b v_a^{-1} = u_{ab}.$$

These follow from

$$\begin{aligned} u_a u_b(0) &= u_a(b) = b + a = u_{b+a}(0), \\ v_a v_b(1) &= v_a(b) = ab = v_{ab}(1), \\ v_a u_b v_a^{-1}(0) &= v_a u_b(0) = v_a(b) = ab = u_{ab}(0). \end{aligned}$$

Next we shall prove the next equations from the first equation and the commutativity of U :

$$\begin{aligned} a + (b + c) &= u_{b+c}(a) = u_c u_b(a) = u_c(a + b) = (a + b) + c, \\ a + b &= u_{a+b}(0) = u_b u_a(0) = u_a u_b(0) = u_a(b) = b + a, \\ a + 0 &= 0 + a = u_a(0) = a, \\ a + u_a^{-1}(0) &= u_a^{-1}(0) + a = u_a(u_a^{-1}(0)) = \varepsilon(0) = 0. \end{aligned}$$

We shall prove the next equations from the second equation for $a, b \in \Delta^*$. For $a = 0$ or $b = 0$, it is easy to prove our equations:

$$\begin{aligned} a(bc) &= v_a(bc) = v_a(v_b(c)) = v_a v_b(c) = v_{ab}(c) = (ab)c, \\ a1 &= v_a(1) = a = \varepsilon(a) = v_1(a) = 1a, \\ av_a^{-1}(1) &= v_a(v_a^{-1}(1)) = \varepsilon(1) = 1. \end{aligned}$$

For $a \in \Delta^*$, $v_a^{-1}(1) \neq 0$ follows from $v_a(0) = 0 \neq 1$ and we can see $v_{v_a^{-1}(1)} = v_a^{-1}$ by $v_{v_a^{-1}(1)}(1) = v_a^{-1}(1)$. Thus we have

$$v_a^{-1}(1)a = v_{v_a^{-1}(1)}(a) = v_a^{-1}(a) = v_a^{-1}(v_a(1)) = 1.$$

The next equation follows from the third equation:

$$a(b + c) = v_a(b + c) = v_a(u_c(b)) = v_a u_c v_a^{-1}(v_a(b)) = u_{ac}(ab) = ab + ac.$$

Thus Δ is a near field by our definition of the sum and the product. Moreover Δ is of characteristic p because $u_{p-1} = u_1^p = \varepsilon = u_0$.

Next we shall show σ is an automorphism of Δ . It is easy to see from the definitions of U and V that

$$\sigma U \sigma^{-1} \subseteq U \quad \text{and} \quad \sigma V \sigma^{-1} \subseteq V.$$

It follows from the definitions of u_a and v_a that

$$\sigma u_a \sigma^{-1} = u_{\sigma(a)} \quad \text{and} \quad \sigma v_b \sigma^{-1} = v_{\sigma(b)}$$

by equations

$$\sigma u_a \sigma^{-1}(0) = \sigma u_a(0) = \sigma(a) = u_{\sigma(a)}(0)$$

and

$$\sigma v_b \sigma^{-1}(1) = \sigma v_b(1) = \sigma(b) = v_{\sigma(b)}(1).$$

Since σ is a permutation on Δ , it follows from the next equations that σ is an automorphism of Δ :

$$u_{\sigma(a+b)} = \sigma u_{a+b} \sigma^{-1} = \sigma u_a \sigma^{-1} \sigma u_b \sigma^{-1} = u_{\sigma(a)} u_{\sigma(b)} = u_{\sigma(a)+\sigma(b)}$$

and

$$v_{\sigma(ab)} = \sigma v_{ab} \sigma^{-1} = \sigma v_a \sigma^{-1} \sigma v_b \sigma^{-1} = v_{\sigma(a)} v_{\sigma(b)} = v_{\sigma(a)\sigma(b)}. \quad \square$$

We can see from Lemma 2 and the classification of finite near fields (see [9]) that Δ is a Dickson near field because Δ has an automorphism of order p where p is the characteristic of Δ .

Lemma 3. *WT is a Frobenius group with kernel T and complement W .*

Proof. We note $W \cap V = \{\varepsilon\}$ since $\sigma(1) = 1$. Let $x = \sigma^k v$ be an element of $WT \setminus W$, where $v \in T$, and let $x^{-1} \sigma^s x = \sigma^t \neq \varepsilon$ be an element of $x^{-1} W x \cap W$. Then we may assume $s = 1$ because the order of σ is p . Thus $x^{-1} W x \cap W$ contains $v^{-1} \sigma v = \sigma^t$. The element $\sigma^{t-1} = v^{-1} \cdot \sigma v \sigma^{-1}$ is contained in $W \cap V = \{\varepsilon\}$. Hence $\sigma v = v \sigma$. Thus $v \in C_V(\sigma) \cap T = \{\varepsilon\}$ and $x = \sigma^k v = \sigma^k$ is contained in W . Therefore we have

$$x^{-1} W x \cap W = \{\varepsilon\} \text{ for } x \in WT \setminus W. \quad \square$$

Lemma 4. $G = TC_G(\sigma)T$.

Proof. Clearly $TC_G(\sigma)T$ contains T and W . Let u_δ be an arbitrary element of $U \setminus \{\varepsilon\}$, where δ is an arbitrary element in $\Delta^* = \Delta \setminus \{0\}$. Then $v_\delta = v_\gamma v_\lambda = v_{\gamma\lambda}$ where $v_\gamma \in T$ and $v_\lambda \in C_V(\sigma)$, namely, $\sigma(\lambda) = \lambda$. Thus $\delta = \gamma\lambda$ and so $u_\delta = v_\gamma u_\lambda v_\gamma^{-1} \in TC_G(\sigma)T$. It follows from $U \subset TC_G(\sigma)T$ that $G = TC_G(\sigma)T$. \square

Lemma 5. $(J(KW)\hat{T}KG)^n \subseteq J(KW)^n \hat{T}KG$, where $\hat{T} = \sum_{t \in T} t$.

Proof. Since T is normal in WV and $G = TC_G(\sigma)T$ by Lemma 4, we can see $s\sigma = \sigma s$ for every $s \in \hat{T}KG\hat{T} = \hat{T}KC_G(\sigma)\hat{T}$. Clearly the result holds for $n = 1$. Assume that the result holds for n . Then using the last assertion, we conclude that

$$\begin{aligned} (J(KW)\hat{T}KG)^{n+1} &\subseteq J(KW)^n \hat{T}KG J(KW)\hat{T}KG \\ &= J(KW)^n \hat{T}KG \hat{T} J(KW)KG \\ &\subseteq J(KW)^{n+1} \hat{T}KG. \end{aligned} \quad \square$$

Theorem. *Let S be a subgroup of V containing T and let p^{s+1} be the order of a Sylow p -subgroup WU of $M = \langle S, W, U \rangle$. Then $t(M) = (s+1)(p-1)+1$.*

Proof. Let v be an arbitrary element of S . Then $v = tc$ where $t \in T$ and $c \in C_V(\sigma)$. Hence we have

$$v\sigma v^{-1} = tc\sigma c^{-1}t^{-1} = t\sigma t^{-1} \in G = \langle T, W, U \rangle.$$

Noting T is normal in V , we have that G is a normal in M and the index $|M : G|$ is relatively prime to p . Therefore we obtain $t(M) = t(G)$ and it is enough to prove in case $M = G$. Since the radical $J(KG)$ contains the kernel $J(KU)KG$ of the natural homomorphism ν of the group algebra KG onto $K(G/U)$, it follows that $\nu(J(KG)) = \nu(J(KW)\hat{T})$ by Lemma 3 and

[2, Theorem 4] and so $J(KG) = J(KW)\hat{T}KG + J(KU)KG$. Since U is a normal and elementary abelian subgroup of order p^s , it is clear that the nilpotency index of $J(KU)KG$ is $s(p-1) + 1$. On the other hand, Lemma 5 shows that $(J(KW)\hat{T}KG)^p = 0$. Since $J(KW)\hat{T}KG$ and $J(KU)KG$ are right ideals of KG , it follows that

$$J(KG)^{(s+1)(p-1)+1} = (J(KW)\hat{T}KG + J(KU)KG)^{p+s(p-1)} = 0,$$

and so $t(G) \leq (s+1)(p-1) + 1$. On the other hand $(s+1)(p-1) + 1 \leq t(G)$ by [7, Theorem 3.3]. This completes the proof. \square

Example 1. Let (q, n) be a Dickson pair where p is a prime and $q = p^r$ for a positive integer r . Then (q^p, n) is also a Dickson pair because $q^p \equiv -1 \pmod{4}$ if and only if $q \equiv -1 \pmod{4}$. Let $F = F_{q^{pn}}$ be a finite field of order q^{pn} and let $D = D_{q^{pn}}$ be a finite Dickson near field defined by the automorphism $\tau: x \rightarrow x^{q^p}$ of F . Then an automorphism $\sigma: x \rightarrow x^{q^n}$ of F is also of D by [9, Satz 18] or [6, Theorem 5] because $p^{rn} = q^n \equiv 1 \pmod{n}$ (see also [6, Theorem 1]).

Let ω be a generator of the multiplicative group F^* and we set $a = \omega^n$, $b = \omega$ in F^* . Then the multiplicative group D^* of D has the structure

$$D^* = \langle a, b \mid a^m = 1, b^n = a^t, bab^{-1} = a^{q^p} \rangle,$$

where $m = \frac{q^{pn}-1}{n}$, $t = \frac{m}{q^p-1}$. Here we use the usual symbol as the product in D for simplicity. Do not confuse with the product in F . We consider some permutations on D :

$$u_c: x \rightarrow x + c \text{ for } c \in D, \quad v_c: x \rightarrow cx \text{ for } c \in D^*.$$

Then we have some relations

$$u_c u_d = u_{d+c}, \quad v_c v_d = v_{cd}, \quad v_c u_d v_c^{-1} = u_{cd}, \quad \sigma u_c \sigma^{-1} = u_{\sigma(c)}, \quad \sigma v_c \sigma^{-1} = v_{\sigma(c)}$$

on u_c, v_c, σ . We set

$$U = \{u_c \mid c \in D\}, \quad V = \{v_c \mid c \in D^*\}, \quad W = \langle \sigma \rangle$$

and

$$T = \{v_c \in V \mid c \in \langle a^{\frac{q^n-1}{n}} \rangle\}.$$

It is easy to see that UV is sharply 2-fold transitive on D ; T is normal in WV and the order of T is $\frac{q^{pn}-1}{q^n-1}$ because products of a and x in D are the same in F . On the other hand, the set $C_V(\sigma)$ is equal to $F_{q^n}^*$ as a set and the order of $C_V(\sigma)$ is $q^n - 1$. Since $\frac{q^{pn}-1}{q^n-1}$ and $q^n - 1$ are relatively prime, we have $V = C_V(\sigma)T$, $C_V(\sigma) \cap T = \{\varepsilon\}$. Let S be a subgroup of V containing T and $M = \langle S, W, U \rangle$. Then $t(M) = (rpn+1)(p-1) + 1$ by Theorem, where p^{rpn+1} is the order of a Sylow p -subgroup WU of M .

If we put $D = F$ for the extreme case $n = 1$, we have the same example as in [3]. \square

Example 2. If $(q, n) \neq (3, 2)$ and p is not a divisor of r , then D_{q^n} has no automorphisms of order p , and so we consider $D_{q^{pn}}$. But D_{3^2} has an automorphism σ of order 3 and we can consider the affine group $G = \langle \sigma, V, U \rangle$ over D_{3^2} where D_{3^2} is a Dickson near fields defined by an automorphism $x \rightarrow x^3$ of $F_{3^2} = F_3[x]/(x^2 + 1) = \{s + ti \mid i^2 = -1, s, t \in F_3\}$, σ is defined by $\sigma(s + ti) = s + t + ti$, and the permutation group U, V are defined as in Example 1. This group G is isomorphic to $Qd(3)$, namely, a group defined by semi-direct product of $F_3^{(2)}$ by $SL(2, 3)$ using the natural action, where $F_3^{(2)}$ is 2-dimensional vector space over F_3 and $SL(2, 3)$ is the special linear group over $F_3^{(2)}$. In this case 3^3 is the order of a Sylow 3-subgroup of G and it is known from [5] that $t(G) = 9 > 7 = 3(3 - 1) + 1$.

This observation is very interesting because quite different objects (see [3] and [5]) are unified on the ground of Dickson near fields. \square

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(Received April 25, 2002)