モノイダルカテゴリーの研究

研究組織

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## 研究経費

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研究発表

学会誌等

1. D.Tambara, Representations of tensor categories with fusion rules of self-duality for abelian groups, Israel Journal of Mathematics, in press.
2. A.Hanaki, M.Miyamoto and D.Tambara, Quantum Galois theory for finite groups, Duke Mathematical Journal 97(1999), 541--544.
3. D.Tambara and S.Yamagami, Tensor categories with fusion rules of self-duality for finite abelian groups, Journal of Algebra 209(1998), 692--707.

口頭発表

1．丹原大介、有限体の乗法群と加法群の表現の圏の変形、第１６回代数的組合せ論シンポジウム、１９９９年６月２５日
2．丹原大介、テンソル圏への有限群作用の双対性、第３２回環論および表現論シンポジウム、１９９９年１０月５日
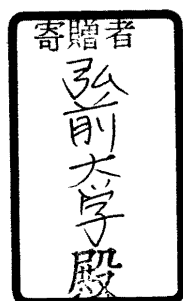
PREFACE

A tensor category is a linear category with operation of tensor product. The category of representations of a group and that of a Hopf algebra are major examples of tensor categories. By analogy with a module over a ring, a module over a tensor category is defined to be a linear category with action of the tensor category. The theme of Chapters I and II is a correspondence between modules over different tensor categories. In Chapter I we relate the category of representations of a finite dimensional semisimple Hopf algebra to the category of representations of its dual Hopf algebra. We give a natural one-to-one correspondence between modules over these two tensor categories. In Chapter II we consider a situation in which a finite group acts on a tensor category. We then have the tensor category of invariant objects and the semi-direct product tensor category, as we make the invariant subring and the skew group ring from a group action on a ring. Using the correspondence of Chapter I, we give a one-to-one correspondence between modules over these two tensor categories as well.

Independently of the first two chapters, Chapter III deals with a special case of the problem of classifying semisimple tensor categories having a prescribed rule of tensor product decomposition. We take the decomposition rule for representations of the semi-direct product of the additive group and the multiplicative group of a finite field. Although we have not reached a complete classification, we give a few nontrivial examples of tensor categories having this rule.

Chapter I and Chapter II are extracted from my papers [1] and [2], respectively. Chapter III is an expanded version of my report [3].

1. A duality for modules over monoidal categories of representations of semisimple Hopf algebras, 1998, in submission.
2. Invariants and semi-direct products for finite group actions on tensor categories, 1999, in submission.
3. Deforming the categories of representations of some semi-direct product groups, in the Proceedings of the 16th Algebraic Combinatorics Symposium, 1999, Fukuoka.

CONTENTS

DUALITY FOR REPRESENTATIONS OF HOPF ALGEBRAS

1. Summary

Let A be a finite dimensional semisimple cosemisimple involutory Hopf algebra over a field k. Let $\mathcal{A}$ be the category of finite dimensional A-modules. As $\mathcal{A}$ is a tensor category, we have a notion of $\mathcal{A}$-modules: A right $\mathcal{A}$-module is a linear category $\mathcal{M}$ equipped with a bilinear functor $\mathcal{M} \times \mathcal{A} \to \mathcal{M}$ and coherent isomorphisms of associativity and unit.

Let B be the dual Hopf algebra of A, and $\mathcal{B}$ the category of finite dimensional B-modules. The main result is that there exists a natural one-to-one correspondence between right $\mathcal{A}$-modules and right $\mathcal{B}$-modules with direct summands.

This is related with the well-known duality theorem ([BM], [NT]) for Hopf algebra actions on algebras.

The correspondence between $\mathcal{A}$-modules and $\mathcal{B}$-modules is given by categorical analogues of Hom and $\otimes$ functors for usual modules. Let $\mathcal{V}$ be the category of k-modules. Then $\mathcal{V}$ becomes an $(\mathcal{A}, \mathcal{B})$-bimodule and also a $(\mathcal{B}, \mathcal{A})$-bimodule. For an $\mathcal{A}$-module $\mathcal{M}$ with direct summands, the corresponding $\mathcal{B}$-module is the category $\operatorname{Hom}_{\mathcal{A}}(\mathcal{V}, \mathcal{M})$ of $\mathcal{A}$-linear functors $\mathcal{V} \to \mathcal{M}$. Alternatively, this $\mathcal{B}$-module is equivalent to the $\mathcal{B}$-module $\mathcal{M} \bar{\otimes}_{\mathcal{A}} \mathcal{V}$, which is obtained by firstly making the tensor product $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{V}$ then adjoining direct summands. An equivalence $\mathcal{M} \bar{\otimes}_{\mathcal{A}} \mathcal{V} \bar{\otimes}_{\mathcal{B}} \mathcal{V} \simeq \mathcal{M}$ is induced by an equivalence of $(\mathcal{A}, \mathcal{A})$-bimodules $\mathcal{V} \bar{\otimes}_{\mathcal{B}} \mathcal{V} \simeq \mathcal{A}$.

In addition, the equivalences $\mathcal{V} \bar{\otimes}_{\mathcal{B}} \mathcal{V} \simeq \mathcal{A}$, $\mathcal{V} \bar{\otimes}_{\mathcal{A}} \mathcal{V} \simeq \mathcal{B}$ can be taken in a coherent way so that the tensor categories $\mathcal{A}$, $\mathcal{B}$, the $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{V}$, and the $(\mathcal{B}, \mathcal{A})$-bimodule $\mathcal{V}$ form a matrix tensor category $\begin{pmatrix} \mathcal{A} & \mathcal{V} \\ \mathcal{V} & \mathcal{B} \end{pmatrix}$.

2. Modules over tensor categories

A k-linear category is a category in which the Hom-sets are k-vector spaces, the compositions are k-bilinear operations and finite direct sums exist. The notion of a k-linear functor $\mathcal{C} \to \mathcal{D}$, and a k-bilinear functor $\mathcal{C} \times \mathcal{C}' \to \mathcal{D}$ for k-linear categories $\mathcal{C}, \mathcal{C}', \mathcal{D}$ will be obvious. Let $\operatorname{Hom}(\mathcal{C}, \mathcal{D})$ denote the category of k-linear functors $\mathcal{C} \to \mathcal{D}$.

Tensor categories. A tensor category over k is a k-linear category $\mathcal{A}$ equipped with a k-bilinear functor $\odot \colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, an object I, and natural isomorphisms

$$\alpha_{X,Y,Z} \colon X \odot (Y \odot Z) \to (X \odot Y) \odot Z,$$
$$\lambda_X \colon X \to I \odot X, \quad \rho_X \colon X \to X \odot I$$

satisfying the identities

$$(\alpha_{X,Y,Z} \odot W)\alpha_{X,Y \odot Z,W}(X \odot \alpha_{Y,Z,W}) = \alpha_{X \odot Y,Z,W}\alpha_{X,Y,Z \odot W}, \qquad \text{(M1)}$$

$$\alpha_{X,I,Y}(X \odot \lambda_Y) = \rho_X \odot Y \qquad \text{(M2)}$$

for all objects X, Y, Z, W in $\mathcal{A}$. See [EK] or [M] for details.

Modules over tensor categories. For a tensor category $\mathcal{A}$, a left $\mathcal{A}$-module is a k-linear category $\mathcal{M}$ equipped with a k-bilinear functor $\odot \colon \mathcal{A} \times \mathcal{M} \to \mathcal{M}$ and natural isomorphisms

$$\alpha_{X,Y,M} \colon X \odot (Y \odot M) \to (X \odot Y) \odot M,$$
$$\lambda_X \colon M \to I \odot M$$

for $X, Y \in \mathcal{A}$, $M \in \mathcal{M}$, satisfying (M1) with (X, Y, Z, W) replaced by (X, Y, Z, M) and (M2) with (X, Y) replaced by (X, M) for all $X, Y, Z \in \mathcal{A}$, $M \in \mathcal{M}$.

A right $\mathcal{A}$-module is similarly defined.

For tensor categories $\mathcal{A}$ and $\mathcal{B}$, an $(\mathcal{A}, \mathcal{B})$-bimodule is a k-linear category $\mathcal{M}$ equipped with k-bilinear functors $\odot \colon \mathcal{A} \times \mathcal{M} \to \mathcal{M}$, $\odot \colon \mathcal{M} \times \mathcal{B} \to \mathcal{M}$, and natural isomorphisms

$$\alpha_{X,Y,M} \colon X \odot (Y \odot M) \to (X \odot Y) \odot M,$$
$$\alpha_{X,M,S} \colon X \odot (M \odot S) \to (X \odot M) \odot S,$$
$$\alpha_{M,S,T} \colon M \odot (S \odot T) \to (M \odot S) \odot T,$$
$$\lambda_M \colon M \to I \odot M, \quad \rho_M \colon M \to M \odot I$$

for $X, Y \in \mathcal{A}$, $M \in \mathcal{M}$, $S, T \in \mathcal{B}$ satisfying (M1) with (X, Y, Z, W) replaced by (X, Y, Z, M), (X, Y, M, S), (X, M, S, T), (M, S, T, U), and (M2) with (X, Y) replaced by (X, M), (M, S) for all $X, Y, Z \in \mathcal{A}$, $M \in \mathcal{M}$, $S, T, U \in \mathcal{B}$.

For left $\mathcal{A}$-modules $\mathcal{M}$ and $\mathcal{N}$, an $\mathcal{A}$-linear functor $(F, \phi) \colon \mathcal{M} \to \mathcal{N}$ consists of a k-linear functor $F \colon \mathcal{M} \to \mathcal{N}$ and natural isomorphisms

$$\phi_{X,M} \colon F(X \odot M) \to X \odot F(M)$$

satisfying the identities

$$\phi_{X \odot Y, M} F(\alpha_{X,Y,M}) = \alpha_{X,Y,F(M)}(X \odot \phi_{Y,M})\phi_{X,Y \odot M},$$
$$\phi_{I,M} F(\lambda_M) = \lambda_{F(M)}$$

for all $X, Y \in \mathcal{A}$, $M \in \mathcal{M}$. We write $(F, \phi) = F$ occasionally.

For $\mathcal{A}$-linear functors $(F, \phi), (F', \phi') \colon \mathcal{M} \to \mathcal{N}$, a morphism $(F, \phi) \to (F', \phi')$ is a natural transformation $\sigma \colon F \to F'$ satisfying

$$\phi'_{X,M} \sigma_{X \odot M} = (X \odot \sigma_M)\phi_{X,M}$$

for all $X \in \mathcal{A}$, $M \in \mathcal{M}$.

With this notion of morphisms, we have the category of $\mathcal{A}$-linear functors $\mathcal{M} \to \mathcal{N}$, denoted by $\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$.

For $\mathcal{A}$-linear functors $(F, \phi) \colon \mathcal{M} \to \mathcal{N}$ and $(G, \psi) \colon \mathcal{N} \to \mathcal{P}$, their composite $(G, \psi) \circ (F, \phi)$ is defined to be the $\mathcal{A}$-linear functor $(G \circ F, \theta) \colon \mathcal{M} \to \mathcal{P}$, where

$$\theta_{X,M} = \psi_{X,F(M)} \circ G(\phi_{X,M}).$$

Thus we have the composition functors

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{N}, \mathcal{P}) \times \mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{P}),$$

which are strictly associative. Also we have the identity $\mathcal{A}$-linear functors $\mathrm{Id}_{\mathcal{M}}$ in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{M})$, which are strictly unital for composition. So the categories $\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$ for all $\mathcal{A}$-modules $\mathcal{M}, \mathcal{N}$ constitute a 2-category, denoted by $\mathcal{A}$-Mod.

An $\mathcal{A}$-linear functor $F: \mathcal{M} \to \mathcal{N}$ is called an equivalence if there are an $\mathcal{A}$-linear functor $G: \mathcal{N} \to \mathcal{M}$ and isomorphisms $F \circ G \cong 1$ in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{N}, \mathcal{N})$, $G \circ F \cong 1$ in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{M})$.

Let $\mathcal{V}$ be the tensor category of finite dimensional vector spaces over k. Any k-linear category $\mathcal{C}$ becomes a left $\mathcal{V}$-module by setting $k^n \otimes X = X^n$, the n-fold direct sum.

Let $\mathcal{L}$ be a $(\mathcal{B}, \mathcal{A})$-bimodule. If $\mathcal{N}$ is a left $\mathcal{B}$-module, the category $\mathrm{Hom}_{\mathcal{B}}(\mathcal{L}, \mathcal{N})$ becomes a left $\mathcal{A}$-module. The action is defined by

$$(X \odot F)(L) = F(L \odot X)$$

for $X \in \mathcal{A}$, $L \in \mathcal{L}$, $F \in \mathrm{Hom}_{\mathcal{B}}(\mathcal{L}, \mathcal{N})$.

Moreover we have a functor

$$\Phi_{\mathcal{N}, \mathcal{N}'}: \mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{N}') \to \mathrm{Hom}_{\mathcal{A}}(\mathrm{Hom}_{\mathcal{B}}(\mathcal{L}, \mathcal{N}), \mathrm{Hom}_{\mathcal{B}}(\mathcal{L}, \mathcal{N}'))$$
$$G \mapsto (F \mapsto G \circ F)$$

for $\mathcal{B}$-modules $\mathcal{N}, \mathcal{N}'$. The functors $\Phi_{\mathcal{N}, \mathcal{N}'}$ preserve horizontal compositions and unit 1-cells.

The 2-functor $\mathrm{Hom}_{\mathcal{B}}(\mathcal{L}, -): \mathcal{B}\text{-Mod} \to \mathcal{A}\text{-Mod}$ consists of the assignment

$$\mathcal{B}\text{-module } \mathcal{N} \mapsto \mathcal{A}\text{-module } \mathrm{Hom}_{\mathcal{B}}(\mathcal{L}, \mathcal{N})$$

and the collection of the functors $\Phi_{\mathcal{N}, \mathcal{N}'}$ for all $\mathcal{B}$-modules $\mathcal{N}, \mathcal{N}'$.

Tensor product of modules. For a right $\mathcal{A}$-module $\mathcal{M}$, a left $\mathcal{A}$-module $\mathcal{N}$, and a k-linear category $\mathcal{L}$, an $\mathcal{A}$-bilinear functor $(F, \alpha): \mathcal{M} \times \mathcal{N} \to \mathcal{L}$ consists of a k-bilinear functor $F: \mathcal{M} \times \mathcal{N} \to \mathcal{L}$ and natural isomorphisms

$$\alpha_{M, X, N}: F(M, X \odot N) \to F(M \odot X, N)$$

satisfying

$$F(\alpha_{M, X, Y}, N)\alpha_{M, X \odot Y, N}F(M, \alpha_{X, Y, N}) = \alpha_{M \odot X, Y, N}\alpha_{M, X, Y \odot N},$$
$$\alpha_{M, I, N}F(M, \lambda_N) = F(\rho_M, N)$$

for all $M \in \mathcal{M}$, $N \in \mathcal{N}$, $X, Y \in \mathcal{A}$.

With an obvious definition of morphisms, we have the category of $\mathcal{A}$-bilinear functors $\mathcal{M} \times \mathcal{N} \to \mathcal{L}$, denoted by $\mathrm{BiHom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}; \mathcal{L})$.

We will construct a k-linear category $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ and an $\mathcal{A}$-bilinear functor $\mathcal{M} \times \mathcal{N} \to \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ inducing an equivalence $\mathrm{Hom}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}, \mathcal{L}) \to \mathrm{BiHom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}; \mathcal{L})$ for any k-linear category $\mathcal{L}$.

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As a k-linear category, $\mathcal{M} \otimes_A \mathcal{N}$ has the following presentation by generators and relations. Objects are finite direct sums of symbols $[M, N]$ for $M \in \mathcal{M}$, $N \in \mathcal{N}$. Generators for morphisms are symbols

$$[f, g] \colon [M, N] \to [M', N']$$

for morphisms $f \colon M \to M'$ in $\mathcal{M}$ and $g \colon N \to N'$ in $\mathcal{N}$, and symbols

$$\alpha_{M,X,N} \colon [M, X \odot N] \to [M \odot X, N]$$
$$\alpha'_{M,X,N} \colon [M \odot X, N] \to [M, X \odot N]$$

for objects $M \in \mathcal{M}$, $X \in \mathcal{A}$, $N \in \mathcal{N}$. Relations among them are
  (i) (linearity)

$$[f + f', g] = [f, g] + [f', g], \quad [f, g + g'] = [f, g] + [f, g']$$
$$[af, g] = a[f, g] = [f, ag]$$

for morphisms $f, f' \colon M \to M'$ in $\mathcal{M}$, $g, g' \colon N \to N'$ in $\mathcal{N}$, and $a \in k$.
 (ii) (functoriality)

$$[f_2, g_2][f_1, g_1] = [f_2 f_1, g_2 g_1]$$

for morphisms $f_1 \colon M_1 \to M_2$, $f_2 \colon M_2 \to M_3$ in $\mathcal{M}$ and $g_1 \colon N_1 \to N_2$, $g_2 \colon N_2 \to N_3$ in $\mathcal{N}$, and

$$[1_M, 1_N] = 1_{[M,N]}.$$

(iii) (isomorphism)

$$\alpha_{M,X,N} \alpha'_{M,X,N} = 1, \quad \alpha'_{M,X,N} \alpha_{M,X,N} = 1.$$

(iv) (naturality)

$$\alpha_{M',X',N'}[f, u \odot g] = [f \odot u, g]\alpha_{M,X,N}$$

for morphisms $f \colon M \to M'$ in $\mathcal{M}$, $u \colon X \to X'$ in $\mathcal{A}$, $g \colon N \to N'$ in $\mathcal{N}$.
 (v) (pentagon and triangle)

$$[\alpha_{M,X,Y}, 1_N]\alpha_{M,X\odot Y,N}[1_M, \alpha_{X,Y,N}] = \alpha_{M\odot X,Y,N}\alpha_{M,X,Y\odot N},$$
$$\alpha_{M,I,N}[1_M, \rho_N] = [\lambda_M, 1_N].$$

The bilinear functor $T \colon \mathcal{M} \times \mathcal{N} \to \mathcal{M} \otimes_A \mathcal{N}$ is defined by

$$T(M, N) = [M, N] \text{ for objects,}$$
$$T(f, g) = [f, g] \text{ for morphisms.}$$

The isomorphisms $\alpha_{M,X,N}$ then give T a structure of an $\mathcal{A}$-bilinear functor.

From this construction, it will be obvious that for any k-linear category $\mathcal{L}$, the functor

$$\mathrm{Hom}(\mathcal{M} \otimes_A \mathcal{N}, \mathcal{L}) \to \mathrm{BiHom}_\mathcal{A}(\mathcal{M}, \mathcal{N}; \mathcal{L})$$
$$G \mapsto G \circ T$$

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is an equivalence.

Let $\mathcal{L}$ be a $(\mathcal{B}, \mathcal{A})$-bimodule. If $\mathcal{M}$ is a left $\mathcal{A}$-module, $\mathcal{L} \otimes_{\mathcal{A}} \mathcal{M}$ becomes a left $\mathcal{B}$-module. The action is defined by

$$S \odot [L, M] = [S \odot L, M]$$

for $S \in \mathcal{B}$, $M \in \mathcal{M}$, $L \in \mathcal{L}$.

Moreover we have a functor

$$\Psi_{\mathcal{M}, \mathcal{M}'} \colon \operatorname{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{M}') \to \operatorname{Hom}_{\mathcal{B}}(\mathcal{L} \otimes_{\mathcal{A}} \mathcal{M}, \mathcal{L} \otimes_{\mathcal{A}} \mathcal{M}')$$
$$G \mapsto ([L, M] \mapsto [L, G(M)])$$

for $\mathcal{A}$-modules $\mathcal{M}, \mathcal{M}'$. The functors $\Psi_{\mathcal{M}, \mathcal{M}'}$ preserve horizontal compositions and unit 1-cells.

The 2-functor $\mathcal{L} \otimes_{\mathcal{A}} - \colon \mathcal{A}\text{-Mod} \to \mathcal{B}\text{-Mod}$ consists of the assignment

$$\mathcal{A}\text{-module } \mathcal{M} \mapsto \mathcal{B}\text{-module } \mathcal{L} \otimes_{\mathcal{A}} \mathcal{M}$$

and the collection of the functors $\Psi_{\mathcal{M}, \mathcal{M}'}$ for all $\mathcal{A}$-modules $\mathcal{M}, \mathcal{M}'$.

We have also an $\mathcal{A}$-linear functor

$$\mathcal{M} \to \operatorname{Hom}_{\mathcal{B}}(\mathcal{L}, \mathcal{L} \otimes_{\mathcal{A}} \mathcal{M})$$
$$M \mapsto (L \mapsto [L, M])$$

for an $\mathcal{A}$-module $\mathcal{M}$, and a $\mathcal{B}$-linear functor

$$\mathcal{N} \otimes_{\mathcal{A}} \operatorname{Hom}_{\mathcal{B}}(\mathcal{L}, \mathcal{N}) \to \mathcal{N}$$
$$[N, F] \mapsto F(N)$$

for a $\mathcal{B}$-module $\mathcal{N}$. These are natural in $\mathcal{M}$ and $\mathcal{N}$, respectively.

Furthermore we have an equivalence

$$\operatorname{Hom}_{\mathcal{B}}(\mathcal{L} \otimes_{\mathcal{A}} \mathcal{M}, \mathcal{N}) \to \operatorname{Hom}_{\mathcal{A}}(\mathcal{M}, \operatorname{Hom}_{\mathcal{B}}(\mathcal{L}, \mathcal{N}))$$
$$F \mapsto \big(M \mapsto (L \mapsto F([L, M])) \big).$$

Bicategories. A bicategory $\mathcal{E}$ consists of a set J, a collection of k-linear categories $\mathcal{E}_{ij}$ for $i, j \in J$, bilinear functors $\odot_{ijk} \colon \mathcal{E}_{ij} \times \mathcal{E}_{jk} \to \mathcal{E}_{ik}$ for $i, j, k \in I$, objects $I_i \in \mathcal{E}_{ii}$ and natural isomorphisms

$$\alpha_{X,Y,Z} \colon X \odot_{ijl} (Y \odot_{jkl} Z) \to (X \odot_{ijk} Y) \odot_{ikl} Z,$$
$$\lambda_X \colon X \to I_i \odot_{iij} X, \quad \rho_X \colon X \to X \odot_{ijj} I_j$$

for $X \in \mathcal{E}_{ij}$, $Y \in \mathcal{E}_{jk}$, $Z \in \mathcal{E}_{kl}$ satisfying identities analogous to (M1) and (M2). See [B].

For a bicategory $\mathcal{E}$, each category $\mathcal{E}_{ii}$ becomes a tensor category and $\mathcal{E}_{ij}$ becomes an $(\mathcal{E}_{ii}, \mathcal{E}_{jj})$-bimodule. Moreover $\odot_{ijk} \colon \mathcal{E}_{ij} \times \mathcal{E}_{jk} \to \mathcal{E}_{ik}$ becomes an $\mathcal{E}_{jj}$-bilinear functor, and hence induces a functor $\mathcal{E}_{ij} \otimes_{\mathcal{E}_{jj}} \mathcal{E}_{jk} \to \mathcal{E}_{ik}$. This in turn becomes an $(\mathcal{E}_{ii}, \mathcal{E}_{kk})$-linear functor.

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Idempotent splitting property. A category $\mathcal{C}$ is said to have direct summands if any idempotent endomorphism $e\colon X \to X$ in $\mathcal{C}$ has a factorization $e = ip$ with $p\colon X \to Y$, $i\colon Y \to X$ and $1_Y = pi$. The envelope $\bar{\mathcal{C}}$ of $\mathcal{C}$ is the category defined as follows. An object of $\bar{\mathcal{C}}$ is pair (X, e) of an object $X \in \mathcal{C}$ and an idempotent $e \in \mathrm{End}\, X$. A morphism $(X, e) \to (X', e')$ is a morphism $f\colon X \to X'$ in $\mathcal{C}$ such that $fe = f = e'f$. Then $\bar{\mathcal{C}}$ has direct summands and the functor $\mathcal{C} \to \bar{\mathcal{C}}\colon X \mapsto (X, 1)$ has the following universality: For any category $\mathcal{D}$ with direct summands, the induced functor

$$\mathrm{Hom}(\bar{\mathcal{C}}, \mathcal{D}) \to \mathrm{Hom}(\mathcal{C}, \mathcal{D})$$

is an equivalence.

For a tensor category $\mathcal{A}$, $\mathcal{A}\text{-Mod}k$ denotes the 2-category consisting of left $\mathcal{A}$-modules with direct summands. If $\mathcal{M}$ is an $\mathcal{A}$-module, then $\bar{\mathcal{M}}$ becomes naturally an $\mathcal{A}$-module. For a right $\mathcal{A}$-module $\mathcal{M}$ and a left $\mathcal{A}$-module $\mathcal{N}$, the envelope of $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ is denoted by $\mathcal{M} \bar{\otimes}_{\mathcal{A}} \mathcal{N}$.

3. The bicategory associated with the dual pair (A, B)

Let A be a finite dimensional semisimple cosemisimple involutory Hopf algebra. Let $B = A^*$ the dual Hopf algebra. Put $\mathcal{A} = A\text{-Mod}$, $\mathcal{B} = B\text{-Mod}$, $\mathcal{V} = k\text{-Mod}$.

In this section we construct a bicategory $\mathcal{E}$, in the sense of Section 2, with indexed set $J = \{1, 2\}$ such that

$$\begin{pmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} \\ \mathcal{E}_{21} & \mathcal{E}_{22} \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \mathcal{V} \\ \mathcal{V} & \mathcal{B} \end{pmatrix}.$$

The canonical pairing between A and B is denoted by $\langle -, - \rangle$. After Sweedler's book [S], the left action $\rightharpoonup$ and the right action $\leftharpoonup$ of A on B are defined by

$$a \rightharpoonup b = \sum b_1 \langle a, b_2 \rangle,$$
$$b \leftharpoonup a = \sum \langle a, b_1 \rangle b_2$$

for $a \in A$, $b \in B$ with $\Delta(b) = \sum b_1 \otimes b_2$, so that

$$\langle a', a \rightharpoonup b \rangle = \langle a'a, b \rangle,$$
$$\langle a', b \leftharpoonup a \rangle = \langle aa', b \rangle.$$

Then the left action $\rightharpoondown$ and the right action $\leftharpoondown$ of A on B are defined by

$$a \rightharpoondown b = b \leftharpoonup S(a),$$
$$b \leftharpoondown a = S(a) \rightharpoonup b.$$

We need to choose linear isomorphisms $A \to B$, $B \to A$ in a special way.

PROPOSITION. *There exist linear isomorphisms* $\phi\colon A \to B$, $\psi\colon B \to A$ *such that*

$$\phi(a'a) = a' \rightharpoondown \phi(a), \qquad \phi(aa') = \phi(a) \leftharpoondown a',$$
$$\phi(b' \rightharpoonup a) = b'\phi(a), \qquad \phi(a \leftharpoonup b') = \phi(a)b',$$
$$\psi(b'b) = b' \rightharpoondown \psi(b), \qquad \psi(bb') = \psi(b) \leftharpoondown b',$$
$$\psi(a' \rightharpoonup b) = a'\psi(b), \qquad \psi(b \leftharpoonup a') = \psi(b)a'$$

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for all $a, a' \in A$, $b, b' \in B$, and that

$$S\psi\phi = 1, \qquad S\phi\psi = 1.$$

Such a pair (ϕ, ψ) is unique modulo relation $(\phi, \psi) \sim (\lambda\phi, \lambda^{-1}\psi)$ for scalars $\lambda \neq 0$.

We fix such a choice of $\phi\colon A \to B$, $\psi\colon B \to A$.

For $X \in \mathcal{A}$, $Y \in \mathcal{B}$ we have maps

$$\lambda_X\colon X \otimes A \to X \otimes A$$
$$x \otimes a \mapsto \sum a_1 x \otimes a_2,$$
$$\rho_X\colon A \otimes X \to A \otimes X$$
$$a \otimes x \mapsto \sum a_1 \otimes a_2 x,$$
$$\beta_{X,Y}\colon X \otimes Y \to X \otimes Y$$
$$x \otimes y \mapsto \sum a_i x \otimes y_i = \sum x_j \otimes b_j y,$$
$$\gamma_Y\colon A \otimes Y \to Y \otimes A$$
$$a \otimes y \mapsto \sum y_i \otimes a a_i,$$

where

$$\Delta(a) = \sum a_1 \otimes a_2, \quad \omega(y) = \sum y_i \otimes a_i, \quad \omega(x) = \sum x_j \otimes b_j$$

and $\omega\colon Y \to Y \otimes A$, $\omega\colon X \to X \otimes B$ are the right comodule structures coming from the left module structures.

These are all bijections with inverses given by

$$\lambda_X^{-1}\colon x \otimes a \mapsto \sum S^{-1}(a_1) x \otimes a_2,$$
$$\rho_X^{-1}\colon a \otimes x \mapsto \sum a_1 \otimes S(a_2) x,$$
$$\beta_{X,Y}^{-1}\colon x \otimes y \mapsto \sum S(a_i) x \otimes y_i = \sum x_j \otimes S(b_j) y,$$
$$\gamma_Y^{-1}\colon y \otimes a \mapsto \sum a S^{-1}(a_i) \otimes y_i.$$

Replacing the roles of A and B, we have similar maps $\lambda_Y, \rho_Y, \beta_{Y,X}, \gamma_X$ for $X \in \mathcal{A}$, $Y \in \mathcal{B}$.

Now we define the bicategory $\mathcal{E}$ as follows. The index set is $\{1, 2\}$. Let

$$\mathcal{E}_{11} = \mathcal{A}, \quad \mathcal{E}_{12} = \mathcal{V},$$
$$\mathcal{E}_{21} = \mathcal{V}, \quad \mathcal{E}_{22} = \mathcal{B}.$$

The composition functors

$$\odot_{ijk}\colon \mathcal{E}_{ij} \times \mathcal{E}_{jk} \to \mathcal{E}_{ik}$$

for $i, j, k = 1, 2$ are given by

$$X \odot_{111} X' = X \otimes X', \qquad Y \odot_{222} Y' = Y \otimes Y',$$
$$X \odot_{112} V = X \otimes V, \qquad Y \odot_{221} V = Y \otimes V,$$
$$V \odot_{211} X = V \otimes X, \qquad V \odot_{122} Y = V \otimes Y,$$
$$V \odot_{121} V' = V \otimes A \otimes V', \qquad V \odot_{212} V' = V \otimes B \otimes V'$$

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for $X, X' \in \mathcal{A}$, $Y, Y' \in \mathcal{B}$, $V, V' \in \mathcal{V}$. Here the module structures of $X \otimes X'$, $Y \otimes Y'$ are the usual ones. In $X \otimes V$, $V \otimes X$, $Y \otimes V$, $V \otimes Y$ the module structures of X, Y are forgotten. In $V \otimes A \otimes V'$, $V \otimes B \otimes V'$, we regard A, B as the left regular modules.

The units $I \in \mathcal{E}_{11}$, $I \in \mathcal{E}_{22}$ are the trivial modules k.

Next we define the natural transformations of associativity

$$\alpha_{ijkl} \colon \odot_{ijl} \circ (1_{\mathcal{E}_{ij}} \times \odot_{jkl}) \to \odot_{ikl} \circ (\odot_{ijk} \times 1_{\mathcal{E}_{kl}}).$$

α_{1111}, α_{2222}, α_{1112}, α_{2111}, α_{2221}, α_{1222} are the identity. α_{1122}, α_{1121}, α_{1211}, α_{1221}, α_{1212} are given by

$$
\begin{array}{ccc}
X \odot (V \odot Y) & \xrightarrow{\alpha_{1122}} & (X \odot V) \odot Y \\
\| & & \| \\
X \otimes V \otimes Y & \xrightarrow[(\beta_{X,Y})]{} & X \otimes V \otimes Y
\end{array}
$$

$$
\begin{array}{ccc}
X \odot (V \odot V') & \xrightarrow{\alpha_{1121}} & (X \odot V) \odot V' \\
\| & & \| \\
X \otimes V \otimes A \otimes V' & \xrightarrow[(\lambda_X^{-1})]{} & X \otimes V \otimes A \otimes V'
\end{array}
$$

$$
\begin{array}{ccc}
V \odot (V' \odot X) & \xrightarrow{\alpha_{1211}} & (V \odot V') \odot X \\
\| & & \| \\
V \otimes A \otimes V' \otimes X & \xrightarrow[(\rho_X)]{} & V \otimes A \otimes V' \otimes X
\end{array}
$$

$$
\begin{array}{ccc}
V \odot (Y \odot V) & \xrightarrow{\alpha_{1221}} & (V \odot Y) \odot V' \\
\| & & \| \\
V \otimes A \otimes Y \otimes V' & \xrightarrow[(\gamma_Y)]{} & V \otimes Y \otimes A \otimes V'
\end{array}
$$

$$
\begin{array}{ccc}
V \odot (V' \odot V'') & \xrightarrow{\alpha_{1212}} & (V \odot V') \odot V'' \\
\| & & \| \\
V \otimes V' \otimes B \otimes V'' & \xrightarrow[(\psi)]{} & V \otimes A \otimes V' \otimes V''
\end{array}
$$

for $X \in \mathcal{A}$, $Y \in \mathcal{B}$, $V, V', V'' \in \mathcal{V}$. Here $(\beta_{X,Y})$, (γ_X), ... stand for the maps induced by $\beta_{X,Y}$, γ_X, ... in an obvious way. The remaining α_{2211}, α_{2212}, α_{2122}, α_{2112}, α_{2121} are defined by replacing A and B, ϕ and ψ.

Finally the natural isomorphisms for unit

$$\lambda_{ij} \colon 1_{\mathcal{E}_{ij}} \to I_i \odot_{iij} (-), \quad \rho_{ij} \colon 1_{\mathcal{E}_{ij}} \to (-) \odot_{ijj} I_j$$

are given by the maps $x \mapsto 1 \otimes x$, $x \mapsto x \otimes 1$.

THEOREM. *The data $\mathcal{E}_{ij}$, $\odot_{ijk}$, I_i, α_{ijkl}, λ_{ij}, ρ_{ij} constitute a bicategory $\mathcal{E}$.*

4. The correspondence between $\mathcal{A}$-modules and $\mathcal{B}$-modules

We keep the notation and the assumptions in the preceding section. Let Modk-$\mathcal{A}$ denote the 2-category of right $\mathcal{A}$-modules with direct summands. We will construct a 2-equivalence between the 2-categories Modk-$\mathcal{A}$ and Modk-$\mathcal{B}$.

Since $\mathcal{E}$ is a bicategory, $\mathcal{E}_{12}$ naturally becomes an $(\mathcal{E}_{11}, \mathcal{E}_{22})$-bimodules. That is, $\mathcal{V}$ becomes an $(\mathcal{A}, \mathcal{B})$-bimodule. And similarly $\mathcal{V}$ becomes a $(\mathcal{B}, \mathcal{A})$-bimodule. The composition $\odot_{121} \colon \mathcal{E}_{12} \times \mathcal{E}_{21} \to \mathcal{E}_{11}$ yields an $(\mathcal{E}_{11}, \mathcal{E}_{11})$-linear functor $\mathcal{E}_{12} \otimes_{\mathcal{E}_{22}} \mathcal{E}_{21} \to \mathcal{E}_{11}$, that is, an $(\mathcal{A}, \mathcal{A})$-linear functor

$$P \colon \mathcal{V} \otimes_{\mathcal{B}} \mathcal{V} \to \mathcal{A}.$$

Similarly, we obtain a $(\mathcal{B}, \mathcal{B})$-linear functor

$$Q \colon \mathcal{V} \otimes_{\mathcal{A}} \mathcal{V} \to \mathcal{B}.$$

As $\mathcal{A}$ has direct summands , P extends to an $(\mathcal{A}, \mathcal{A})$-linear functor

$$\bar{P} \colon \mathcal{V} \bar{\otimes}_{\mathcal{B}} \mathcal{V} \to \mathcal{A}.$$

And similarly we obtain a $(\mathcal{B}, \mathcal{B})$-linear functor

$$\bar{Q} \colon \mathcal{V} \bar{\otimes}_{\mathcal{A}} \mathcal{V} \to \mathcal{B}.$$

As $\mathcal{V}$ is an $(\mathcal{A}, \mathcal{B})$-bimodule, if $\mathcal{M}$ is a right $\mathcal{A}$-module, then $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{V}$ becomes a right $\mathcal{B}$-module, and its envelope $\mathcal{M} \bar{\otimes}_{\mathcal{A}} \mathcal{V}$ becomes a right $\mathcal{B}$-module with direct summands. Similarly, if $\mathcal{N}$ is a right $\mathcal{B}$-module, then $\mathcal{N} \bar{\otimes}_{\mathcal{B}} \mathcal{V}$ becomes a right $\mathcal{A}$-module with direct summands.

For a right $\mathcal{A}$-module $\mathcal{M}$ with direct summands, the functor $\bar{P}$ induces an $\mathcal{A}$-linear functor

$$P_{\mathcal{M}}^{\sharp} \colon (\mathcal{M} \bar{\otimes}_{\mathcal{A}} \mathcal{V}) \bar{\otimes}_{\mathcal{B}} \mathcal{V} \simeq \mathcal{M} \bar{\otimes}_{\mathcal{A}} (\mathcal{V} \bar{\otimes}_{\mathcal{B}} \mathcal{V}) \to \mathcal{M} \bar{\otimes}_{\mathcal{A}} \mathcal{A} \simeq \mathcal{M},$$

and for a right $\mathcal{B}$-module $\mathcal{N}$ with direct summands, $\bar{Q}$ similarly induces a $\mathcal{B}$-linear functor

$$Q_{\mathcal{N}}^{\sharp} \colon (\mathcal{N} \bar{\otimes}_{\mathcal{B}} \mathcal{V}) \bar{\otimes}_{\mathcal{A}} \mathcal{V} \to \mathcal{N}.$$

THEOREM. *For any right $\mathcal{A}$-module $\mathcal{M}$ with direct summands, $P_{\mathcal{M}}^{\sharp}$ is an equivalence of $\mathcal{A}$-modules.*

And $Q_{\mathcal{N}}^{\sharp}$ is an equivalence of $\mathcal{B}$-modules as well. To put it shortly, the 2-functors

$$\text{Modk-}\mathcal{A} \; \underset{-\bar{\otimes}_{\mathcal{B}} \mathcal{V}}{\overset{-\bar{\otimes}_{\mathcal{A}} \mathcal{V}}{\rightleftarrows}} \; \text{Modk-}\mathcal{B}$$

are quasi-inverse to each other.

The theorem follows from

PROPOSITION. *The functor $\bar{P} \colon \mathcal{V} \bar{\otimes}_{\mathcal{B}} \mathcal{V} \to \mathcal{A}$ is an equivalence of $(\mathcal{A}, \mathcal{A})$-bimodules.*

By adjoint this implies also

PROPOSITION. *For any right $\mathcal{A}$-module $\mathcal{M}$ with direct summands, we have an equivalence of right $\mathcal{B}$-modules*

$$\mathcal{M} \bar{\otimes}_{\mathcal{A}} \mathcal{V} \to \operatorname{Hom}_{\mathcal{A}}(\mathcal{V}, \mathcal{M}).$$

5. Duality for Hopf algebra actions

In this section we relate the correspondence between category modules in Section 4 with the duality of Hopf algebra actions on algebras due to Blattner and Montgomery ([BM]). In the beginning we only assume that A is a finite dimensional Hopf algebra and B is the dual of A. For a left A-module algebra R with action written as

$$A \times R \to R \colon (a, r) \mapsto a \triangleright r,$$

the smash product $R \# A$ is the algebra with underlying space $R \otimes A$ and multiplication

$$(r \otimes a)(r' \otimes a') = \sum r(a_1 \triangleright r') \otimes a_2 a',$$

where $\Delta(a) = \sum a_1 \otimes a_2$.

A left $R \# A$-module is thought of as a vector space M with two structures of an R-module and an A-module such that the R-module structure map $R \otimes M \to M$ is A-linear.

Here are several facts whose verifications are straightforward.

(1) It is known that $R \# A$ has a structure of a B-module algebra. The action $\triangleright$ of B on $R \# A$ is given by

$$b \triangleright (r \otimes a) = r \otimes (b \rightharpoonup a).$$

(2) If R is a left A-module algebra, then the category $R \# A$-Mod becomes a right module over A-Mod. The action $\odot \colon R \# A\text{-Mod} \times A\text{-Mod} \to R \# A\text{-Mod}$ is defined as follows: For an $R \# A$-module V and an A-module X, we set

$$V \odot X = V \otimes X$$

on which R and A act by

$$r(v \otimes x) = rv \otimes x,$$
$$a(v \otimes x) = \sum a_1 v \otimes a_2 x$$

for $r \in R$, $a \in A$. With this action $V \otimes X$ becomes an $R \# A$-module. The associativity isomorphism $V \odot (X \otimes X') \to (V \odot X) \odot X'$ is the identity on $V \otimes X \otimes X'$.

(3) If R is a left A-module algebra, then the category R-Mod becomes a right module over B-Mod. Indeed, for an R-module V and a B-module Y, we set

$$V \odot Y = V \otimes Y$$

on which R acts by

$$r(v \otimes y) = \sum (a_i \triangleright r) v \otimes y_i.$$

Here $\omega(y) = \sum y_i \otimes a_i$ and $\omega \colon Y \to Y \otimes A$ is the A-comodule structure corresponding to the B-module structure on Y. The associativity isomorphism $V \odot (Y \otimes Y') \to (V \odot Y) \odot Y'$ is the identity map on $V \otimes Y \otimes Y'$.

(4) The action of A-Mod on $R \# A$-Mod in (2) and the one in (3) with $R \# A$ regarded as a B-module algebra as in (1) coincide.

Let R be a B-module algebra. Put $\mathcal{A} = A$-Mod, $\mathcal{B} = B$-Mod. By (2), R-Mod is a right $\mathcal{A}$-module.

Assume that A is semisimple, cosemisimple, and involutory. The 2-functor

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{V}, -)\colon \mathrm{Modk\text{-}}\mathcal{A} \to \mathrm{Modk\text{-}}\mathcal{B}$$

takes the $\mathcal{A}$-module R-Mod to the $\mathcal{B}$-module $R\#B$-Mod. In view of (4), the quasi-inverse

$$\mathrm{Hom}_{\mathcal{B}}(\mathcal{V}, -)\colon \mathrm{Modk\text{-}}\mathcal{B} \to \mathrm{Modk\text{-}}\mathcal{A}$$

takes $R\#B$-Mod to $(R\#B)\#A$-Mod. Thus we have an equivalence of $\mathcal{A}$-modules

$$R\text{-Mod} \simeq (R\#B)\#A\text{-Mod}.$$

This explains the Morita equivalence between R and $(R\#B)\#A$ in the duality theorem for Hopf algebra actions.

DUALITY FOR FINITE GROUP
ACTIONS ON TENSOR CATEGORIES

1. Summary

If a group G acts on a ring S, we have the ring of G-invariants S^G and the skew group ring $S[G]$. We are here concerned with analogous constructions for a tensor category in place of a ring. Suppose that G acts on a tensor category $\mathcal{C}$ over a field k. This means that for each $\sigma \in G$, a tensor functor $\sigma_* \colon \mathcal{C} \to \mathcal{C}$ is given and for each $\sigma, \tau \in G$, a tensor isomorphism $\sigma_* \circ \tau_* \cong (\sigma\tau)_*$ is given in a coherent way. The tensor category $\mathcal{C}^G$ consists of objects C of $\mathcal{C}$ equipped with isomorphisms $\sigma_* C \cong C$ satisfying certain coherence conditions. The tensor category $\mathcal{C}[G]$ is just the product $\bigoplus_{\sigma \in G} \mathcal{C}$ as a category, whose objects are expressed as $\bigoplus_{\sigma \in G}(C_\sigma, \sigma)$ with $C_\sigma \in \mathcal{C}$, and the tensor product in $\mathcal{C}[G]$ is defined by $(C, \sigma) \otimes (D, \tau) = (C \otimes \sigma_* D, \sigma\tau)$.

For a tensor category $\mathcal{A}$, an $\mathcal{A}$-module means a category with associative action of $\mathcal{A}$. We assume here categories have direct sums and direct summands.

Our result is that if G is finite and $k[G]$ is semi-simple, then $\mathcal{C}^G$-modules and $\mathcal{C}[G]$-modules are in one-to-one correspondence. It is given by assigning to a $\mathcal{C}[G]$-module $\mathcal{X}$ the $\mathcal{C}^G$-module $\mathcal{X}^G$ of G-invariant objects of $\mathcal{X}$.

This is a simple consequence of the one-to-one correspondence of Chapter I between modules over the tensor category of $k[G]$-modules and modules over the tensor category of $k[G]^*$-modules, where $k[G]^*$ is the dual of the group algebra.

2. Group actions on tensor categories

An action of a group G on a k-category $\mathcal{X}$ consists of data

- functors $\sigma_* \colon \mathcal{X} \to \mathcal{X}$ for all $\sigma \in G$
- isomorphisms $\phi(\sigma, \tau) \colon (\sigma\tau)_* \to \sigma_* \circ \tau_*$ for all $\sigma, \tau \in G$
- an isomorphism $\nu \colon 1_* \to \mathrm{Id}_{\mathcal{X}}$

which make the the following diagrams commutative for all $\sigma, \tau, \rho \in G$ and $X \in \mathcal{X}$.

$$
\begin{array}{ccc}
(\sigma\tau\rho)_* X & \xrightarrow{\phi(\sigma\tau,\rho)x} & (\sigma\tau)_* \rho_* X \\
{\scriptstyle \phi(\sigma,\tau\rho)x} \downarrow & & \downarrow {\scriptstyle \phi(\sigma,\tau)_{\rho_* x}} \\
\sigma_*(\tau\rho)_* X & \xrightarrow[\sigma_*(\phi(\tau,\rho)x)]{} & \sigma_* \tau_* \rho_* X
\end{array}
\qquad (1)
$$

$$
1_* X \underset{1_*(\nu_X)}{\overset{\phi(1,1)x}{\rightleftarrows}} 1_* 1_* X
\qquad (2)
$$

$$
1_* X \underset{\nu_{1_* x}}{\overset{\phi(1,1)x}{\rightleftarrows}} 1_* 1_* X
\qquad (3)
$$

Here commutativity of the last two diagrams means that the opposite arrows are inverse to each other.

Let $\mathcal{X}$, $\mathcal{Y}$ be categories with G-action. A G-linear functor $\mathcal{X} \to \mathcal{Y}$ consists of

- a k-linear functor $L \colon \mathcal{X} \to \mathcal{Y}$
- isomorphisms $\eta(\sigma) \colon L \circ \sigma_* \to \sigma_* \circ L$ for all $\sigma \in G$

making the following diagram commutative for all $\sigma, \tau \in G$ and $X \in \mathcal{X}$.

$$
\begin{array}{ccc}
L((\sigma\tau)_* X) & \xrightarrow{\;\eta(\sigma\tau)_X\;} & (\sigma\tau)_* L(X) \\
L(\phi(\sigma,\tau)_X) \downarrow & & \downarrow \phi(\sigma,\tau)_{L(X)} \\
L(\sigma_* \tau_* X) \xrightarrow[\eta(\sigma)_{\tau_* X}]{} \sigma_* L(\tau_* X) \xrightarrow[\sigma_* \eta(\tau)_X]{} & & \sigma_* \tau_* L(X)
\end{array}
\tag{4}
$$

Let $\mathcal{X}$ be a category with G-action. The category of G-invariants in $\mathcal{X}$, denoted by $\mathcal{X}^G$, is a k-category defined as follows. An objects of $\mathcal{X}^G$ is a pair (X, f), where X is an object of $\mathcal{X}$ and f is a family of isomorphisms $f(\sigma) \colon \sigma_* X \to X$ for all $\sigma \in G$ making the following diagram commutative for all $\sigma, \tau \in G$.

$$
\begin{array}{ccc}
(\sigma\tau)_* X & \xrightarrow{\;f_{\sigma\tau}\;} & X \\
\phi(\sigma,\tau)_X \downarrow & & \uparrow f_\sigma \\
\sigma_* \tau_* X & \xrightarrow[\sigma_*(f_\tau)]{} & \sigma_* X
\end{array}
\tag{5}
$$

A morphism $(X, f) \to (X', f')$ in $\mathcal{X}^G$ is a morphism $u \colon X \to X'$ in $\mathcal{X}$ such that

$$
f'(\sigma) \circ \sigma_* u = u \circ f(\sigma)
$$

for all $\sigma \in G$.

EXAMPLE 1. Let G act on the category $\mathcal{V}$ of vector spaces trivially. This means that all σ_*, $\phi(\sigma, \tau)$, ν are the identities. Then $\mathcal{V}^G$ is the category of $k[G]$-modules.

Let $\mathcal{C}$ be a tensor category with tensor product $(A, B) \mapsto A.B$, unit object I, associativity isomorphisms $\alpha_{A,B,C} \colon (A.B).C \to A.(B.C)$, and unit isomorphisms $\lambda_A \colon I.A \to A$, $\rho_A \colon A.I \to A$

An action of G on the tensor category $\mathcal{C}$ means an action of G on the k-category $\mathcal{C}$ preserving the tensor structure. Namely it consists of data

- tensor functors $\sigma_* \colon \mathcal{C} \to \mathcal{C}$ for all $\sigma \in G$
- isomorphisms $\phi(\sigma, \tau) \colon (\sigma\tau)_* \to \sigma_* \circ \tau_*$ of tensor functors for all $\sigma, \tau \in G$
- an isomorphism $\nu \colon 1_* \to \mathrm{Id}_\mathcal{C}$ of tensor functors

making the diagrams (1), (2), (3) commutative with obvious change of letters. We also use the word G-tensor category for tensor category with G-action.

By the definition of a tensor functor, the above σ_* consists of

- a functor $\sigma_* \colon \mathcal{C} \to \mathcal{C}$
- natural isomorphisms $\psi(\sigma)_{A,B} \colon \sigma_* A . \sigma_* B \to \sigma_*(A.B)$ for all $A, B \in \mathcal{C}$
- an isomorphism $\iota(\sigma) \colon I \to \sigma_* I$

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making the following diagrams commutative for all $A, B, C \in \mathcal{C}$.

$$
\begin{array}{ccc}
(\sigma_* A.\sigma_* B).\sigma_* C & \xrightarrow{\alpha_{\sigma_* A, \sigma_* B, \sigma_* C}} & \sigma_* A.(\sigma_* B.\sigma_* C) \\
\psi(\sigma)_{A,B}.\sigma_* C \downarrow & & \downarrow \sigma_* A.\psi(\sigma)_{B,C} \\
\sigma_*(A.B).\sigma_* C & & \sigma_* A.\sigma_*(B.C) \\
\psi(\sigma)_{A.B,C} \downarrow & & \downarrow \psi(\sigma)_{A,B.C} \\
\sigma_*((A.B).C) & \xrightarrow[\sigma_*(\alpha_{A,B,C})]{} & \sigma_*(A.(B.C))
\end{array}
\qquad (6)
$$

$$
\begin{array}{ccc}
I.I & \xrightarrow{\lambda_I} & I \\
\iota(\sigma).\iota(\sigma) \downarrow & & \downarrow \iota(\sigma) \\
\sigma_* I.\sigma_* I \xrightarrow[\psi(\sigma)_{I,I}]{} & \sigma_*(I.I) \xrightarrow[\sigma_*(\lambda_I)]{} & \sigma_* I
\end{array}
\qquad (7)
$$

The requirement that $\phi(\sigma, \tau)$ is a morphism of tensor functors means that the following diagram is commutative for all $A, B \in \mathcal{C}$.

$$
\begin{array}{ccc}
(\sigma\tau)_* A.(\sigma\tau)_* B & \xrightarrow{\phi(\sigma,\tau)_A.\phi(\sigma,\tau)_B} & \sigma_* \tau_* A.\sigma_* \tau_* B \\
& & \downarrow \psi(\sigma)_{\tau_* A, \tau_* B} \\
\psi(\sigma\tau)_{A,B} \downarrow & & \sigma_*(\tau_* A.\tau_* B) \\
& & \downarrow \sigma_*(\psi(\tau)_{A,B}) \\
(\sigma\tau)_*(A.B) & \xrightarrow[\phi(\sigma,\tau)_{A.B}]{} & \sigma_* \tau_*(A.B)
\end{array}
\qquad (8)
$$

In the presence of the commutativity of (3) and (8), $\nu\colon 1_* \to \mathrm{Id}_\mathcal{C}$ is automatically a morphism of tensor functors. Thus we could say that a G-action on the tensor category $\mathcal{C}$ consists of the data σ_*, $\phi(\sigma, \tau)$, ν, $\psi(\sigma)$, $\iota(\sigma)$ making the diagrams of (1), (2), (3), (6), (7), (8) commutative.

Let $\mathcal{C}$ be a G-tensor category. The category $\mathcal{C}^G$ becomes a tensor category as follows. The tensor product is defined by

$$
(A, f).(B, g) = (A.B, h) \qquad (9)
$$

where

$$
h(\sigma) = f(\sigma).g(\sigma) \circ \psi(\sigma)_{A,B}^{-1}. \qquad (10)
$$

The unit object is (I, ι^{-1}). The associativity and unit isomorphisms are inherited from $\mathcal{C}$.

We now construct another tensor category $\mathcal{C}[G]$ from a G-tensor category $\mathcal{C}$. We set $\mathcal{C}[G] = \bigoplus_{\sigma \in G} \mathcal{C}$ as categories. So an object of $\mathcal{C}[G]$ is expressed as $\bigoplus_{\sigma \in G}(A_\sigma, \sigma)$ with $A_\sigma \in \mathcal{C}$, and a morphism from $\bigoplus_{\sigma \in G}(A_\sigma, \sigma)$ to $\bigoplus_{\sigma \in G}(B_\sigma, \sigma)$ is expressed as $\bigoplus_{\sigma \in G}(f_\sigma, \sigma)$ with $f_\sigma\colon A_\sigma \to B_\sigma$ a morphism in $\mathcal{C}$. The tensor product operation in $\mathcal{C}[G]$ is defined by

$$
\begin{aligned}
(A, \sigma).(B, \tau) &= (A.\sigma_* B, \sigma\tau) \quad \text{for objects,} \\
(f, \sigma).(g, \tau) &= (f.\sigma_* g, \sigma\tau) \quad \text{for morphisms.}
\end{aligned}
$$

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The unit object is $(I, 1)$. The associativity is given by

$$((A, \sigma).(B, \tau)).(C, \rho) = (A.\sigma_* B, \sigma\tau).(C, \rho) = ((A.\sigma_* B).(\sigma\tau)_* C, \sigma\tau\rho)$$

$$\alpha_{(A,\sigma),(B,\tau),(C,\rho)} \downarrow \qquad\qquad\qquad \downarrow (\alpha(A,\sigma,B,\tau,C), \sigma\tau\rho)$$

$$(A, \sigma).((B, \tau).(C, \rho)) = (A, \sigma).(B.\tau_* C, \tau\rho) = (A.\sigma_*(B.\tau_* C)), \sigma\tau\rho)$$

where $\alpha(A, \sigma, B, \tau, C)$ is the composite

$$(A.\sigma_* B).(\sigma\tau)_* C$$

$$\downarrow (A.\sigma_* B).\phi(\sigma,\tau)_C$$

$$(A.\sigma_* B).\sigma_* \tau_* C$$

$$\downarrow \alpha_{A,\sigma_* B, \sigma_* \tau_* C}$$

$$A.(\sigma_* B.\sigma_* \tau_* C)$$

$$\downarrow A.\psi(\sigma)_{B, \tau_* C}$$

$$A.\sigma_*(B.\tau_* C).$$

The left unitality

$$\lambda_{(A,\sigma)} \colon (I, 1).(A, \sigma) = (I.1_* A, \sigma) \to (A, \sigma)$$

is given by

$$I.1_* A \xrightarrow{I.\nu_A} I.A \xrightarrow{\lambda_A} A.$$

The right unitality

$$\rho_{(A,\sigma)} \colon (A, \sigma).(I, 1) = (A.\sigma_* I, \sigma) \to (A, \sigma)$$

is given by

$$A.\sigma_* I \xrightarrow{A.\iota(\sigma)^{-1}} A.I \xrightarrow{\rho_A} A.$$

These data satisfy the axiom of a tensor category.

EXAMPLE 2. With respect to the trivial action of G on $\mathcal{V}$, we have the tensor category $\mathcal{V}[G]$. Objects are of the form $\bigoplus_{\sigma \in G}(V_\sigma, \sigma)$ with $V_\sigma \in \mathcal{V}$. The tensor product is given by

$$(V, \sigma).(W, \tau) = (V \otimes W, \sigma\tau).$$

Thus $\mathcal{V}[G]$ is the category of G-graded vector spaces, or the category of $k[G]^*$-modules when G is finite.

EXAMPLE 3. Suppose G acts on a group A. Then the action of G on the tensor category $\mathcal{V}[A]$ is induced. We have obviously $\mathcal{V}[A][G] = \mathcal{V}[A \rtimes G]$.

Let $\mathcal{C}$ be a G-tensor category. We may view a $\mathcal{C}[G]$-module as a category having actions of $\mathcal{C}$ and G in a compatible way.

EXAMPLE 4. $\mathcal{C}$ itself is a $\mathcal{C}[G]$-module: $(C, \sigma).C' = C.\sigma_* C'$.

EXAMPLE 5. A $\mathcal{V}[G]$-module is nothing but a k-category with G-action: $(k, \sigma).X = \sigma_* X$.

If $\mathcal{X}$ is a $\mathcal{C}[G]$-module, $\mathcal{X}^G$ becomes a $\mathcal{C}^G$-module by a similar action to (9), (10).

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3. $\mathcal{V}^G$-modules and $\mathcal{V}[G]$-modules

Hereafter we assume G is a finite group and the characteristic of k does not divide $|G|$. We denote the category of finite dimensional $k[G]$-modules by $\mathcal{V}^G$, and the category of finite dimensional $k[G]^*$-modules by $\mathcal{V}[G]$.

We make $\mathcal{V}$ into a $(\mathcal{V}[G], \mathcal{V}^G)$-bimodule. The action of objects are given by

$$Y.V = Y \otimes V, \quad V.X = V \otimes X$$

for $X \in \mathcal{V}^G$, $Y \in \mathcal{V}[G]$, $V \in \mathcal{V}$. The associativity of actions

$$(Y.Y').V \to Y.(Y'.V), \quad (V.X).X' \to V.(X.X')$$

are the identity maps, while

$$(Y.V).X \to Y.(V.X)$$

is the map

$$(y, \tau) \otimes v \otimes x \mapsto (y, \tau) \otimes v \otimes \tau^{-1}x,$$

where $x \in X$, $v \in V$, $\tau \in G$, and (y, τ) is an element in the τ-component of the G-graded space Y.

The duality theorem of Chapter I in the case of a group algebra is as follows.

THEOREM. *The 2-functors*

$$\mathcal{V}^G\text{-Mod}k \underset{\mathrm{Hom}_{\mathcal{V}[G]}(\mathcal{V}, -)}{\overset{\mathcal{V} \bar{\otimes}_{\mathcal{V}^G} -}{\rightleftarrows}} \mathcal{V}[G]\text{-Mod}k$$

are quasi-inverse to each other through the adjunction.

In this situation we also say the pair $(\mathcal{V} \otimes_{\mathcal{V}^G} -, \mathrm{Hom}_{\mathcal{V}[G]}(\mathcal{V}, -))$ is a 2-equivalence. The 2-equivalence amounts to the following:

(i) For every $\mathcal{V}^G$-module $\mathcal{X}$ with direct summands there exist a $\mathcal{V}[G]$-module $\mathcal{Y}$ with direct summands and an equivalence $\mathcal{X} \to \mathrm{Hom}_{\mathcal{V}[G]}(\mathcal{V}, \mathcal{Y})$ of $\mathcal{V}^G$-modules.

(ii) For $\mathcal{V}[G]$-modules $\mathcal{Y}, \mathcal{Y}'$ with direct summands, the functor

$$\mathrm{Hom}_{\mathcal{V}[G]}(\mathcal{Y}, \mathcal{Y}') \to \mathrm{Hom}_{\mathcal{V}^G}(\mathrm{Hom}_{\mathcal{V}[G]}(\mathcal{V}, \mathcal{Y}), \mathrm{Hom}_{\mathcal{V}[G]}(\mathcal{V}, \mathcal{Y}'))$$

is an equivalence.

Note that $\mathcal{V}[G]$-modules are just k-categories with G-action. We have also

PROPOSITION. *For any $\mathcal{V}[G]$-module $\mathcal{X}$, we have an equivalence of $\mathcal{V}^G$-modules*

$$\mathcal{X}^G \simeq \mathrm{Hom}_{\mathcal{V}[G]}(\mathcal{V}, \mathcal{X}).$$

4. $\mathcal{C}^G$-modules and $\mathcal{C}[G]$-modules

Let $\mathcal{C}$ be a tensor category with G-action.

THEOREM. *The 2-functors*

$$\mathcal{C}^G\text{-Modk} \quad \overset{C\bar{\otimes}_{C^G}-}{\underset{(-)^G}{\rightleftarrows}} \quad \mathcal{C}[G]\text{-Modk}$$

are quasi-inverse to each other.

Here if $\mathcal{X}$ is a $\mathcal{C}[G]$-module, then $\mathcal{X}^G$ becomes a $\mathcal{C}^G$-module as noted in Section 3. Also in the tensor product $\mathcal{C}\bar{\otimes}_{\mathcal{C}^G}-$, $\mathcal{C}$ is viewed as a $(\mathcal{C}[G],\mathcal{C}^G)$-bimodule in which the left action of $\mathcal{C}[G]$ on $\mathcal{C}$ is the standard one (Example 4), the right action of $\mathcal{C}^G$ on $\mathcal{C}$ comes from the forgetful functor $\mathcal{C}^G \to \mathcal{C}$, and the associativity

$$((X,\sigma).Y).(Z,f) \to (X,\sigma).(Y.(Z,f))$$

for $(X,\sigma) \in \mathcal{C}[G]$, $Y \in \mathcal{C}$, $(Z,f) \in \mathcal{C}^G$ is given by

$$(X.\sigma_*Y).Z \overset{\alpha_{X,\sigma_*Y,Z}}{\longrightarrow} X.(\sigma_*Y.Z) \overset{X.(\sigma_*Y.f(\sigma)^{-1})}{\longrightarrow} X.(\sigma_*Y.\sigma_*Z) \overset{X.\psi(\sigma)_{Y,Z}}{\longrightarrow} X.\sigma_*(Y.Z).$$

5. Modules over group tensor categories

In this section we describe modules over a 3-cocycle deformation of $\mathcal{V}[G]$.

For $\sigma \in G$ we write the object (k,σ) of $\mathcal{V}[G]$ simply as σ. Let $w\colon G^3 \to k^\times$ be a 3-cocycle. We have the tensor category $\mathcal{V}[G,w]$ whose underlying k-category, tensor product and unit object are the same as those of $\mathcal{V}[G]$, and whose associativity and unit isomorphisms are given by

$$\alpha_{\sigma,\tau,\rho} = w(\sigma,\tau,\rho)1_{\sigma\tau\rho}$$
$$\lambda_\sigma = w(1,1,\sigma)^{-1}1_\sigma$$
$$\rho_\sigma = w(\sigma,1,1)1_\sigma$$

for $\sigma,\tau,\rho \in G$. We call $\mathcal{V}[G,w]$ the group tensor category of the pair (G,w).

Analogously to the identification of a $\mathcal{V}[G]$-module with a category with G-action, a $\mathcal{V}[G,w]$-module is thought of as a k-category equipped with σ_*, $\phi(\sigma,\tau)$, ν satisfying the commutativity of the diagrams

$$
\begin{array}{ccc}
(\sigma(\tau\rho))_*X & \overset{w(\sigma,\tau,\rho)1}{\longleftarrow} & ((\sigma\tau)\rho)_*X \\
& & \downarrow{\phi(\sigma\tau,\rho)x} \\
\phi(\sigma,\tau\rho)x \downarrow & & (\sigma\tau)_*\rho_*X \\
& & \downarrow{\phi(\sigma,\tau)_{\rho_*x}} \\
\sigma_*(\tau\rho)_*X & \underset{\sigma_*(\phi(\tau,\rho)x)}{\longrightarrow} & \sigma_*\tau_*\rho_*X
\end{array}
$$

(instead of (1)), (2) and (3).

A k-category that is equivalent to a finite direct sum of $\mathcal{V}$ is called a 2-vector space.

All $\mathcal{V}[G, w]$-modules that are 2-vector spaces as categories can be obtained as follows. Let X be a finite G-set and $v: G \times G \times X \to k^\times$ a map satisfying

$$w(\sigma, \tau, \rho) = \frac{v(\sigma\tau, \rho; x)v(\sigma, \tau; \rho x)}{v(\tau, \rho; x)v(\sigma, \tau\rho; x)}$$

for $\sigma, \tau, \rho \in G$, $x \in X$. If v is viewed as a map $G \times G \to \mathrm{Map}(X, k^\times)$, the equations read as

$$i_*(w) = \partial v^{-1}$$

in $\mathrm{Map}(G^3, \mathrm{Map}(X, k^\times))$, where ∂ is the coboundary operator for the group G and i_* is the map induced by the embedding $i: k^\times \to \mathrm{Map}(X, k^\times)$. Let $\mathcal{V}[X]$ denote $\bigoplus_{x \in X} \mathcal{V}$, the category of X-graded vector spaces. We may regard an element $x \in X$ as a simple object of $\mathcal{V}[X]$. The action of $\mathcal{V}[G, w]$ on $\mathcal{V}[X]$ is then defined by

$$\sigma_* x = \sigma x$$
$$\phi(\sigma, \tau)_x = v(\sigma, \tau; x)1_{\sigma\tau x}$$
$$\nu_x = \frac{1}{v(1, 1; x)}1_x$$

for $\sigma, \tau \in G$, $x \in X$. We denote by $\mathcal{V}[X, v]$ the $\mathcal{V}[G, w]$-module obtained in this way. Given two pairs (X, v), (X', v') as above, the $\mathcal{V}[G, w]$-modules $\mathcal{V}[X, v]$ and $\mathcal{V}[X', v']$ are equivalent if and only if there exists an isomorphism $f: X \to X'$ of G-sets such that $f^*(v')$ and v are cohomologue in the group $\mathrm{Map}(G^2, \mathrm{Map}(X, k^\times))$. Thus the equivalence class of a $\mathcal{V}[G, w]$-module which is a 2-vector space bijectively corresponds to the isomorphism class of a pair $(X, [v])$ of a finite G-set X and an element $[v]$ in the quotient set

$$\frac{\{v \in \mathrm{Map}(G^2, \mathrm{Map}(X, k^\times)) \mid \partial v = i_*(w)^{-1}\}}{\{\partial t \mid t \in \mathrm{Map}(G, \mathrm{Map}(X, k^\times))\}}.$$

Here the group in the denominator acts on the set in the numerator by translation. Note that the quotient is either an empty set or a regular $H^2(G, \mathrm{Map}(X, k^\times))$-set.

Let $w = 1$. Then $\mathcal{V}[G, w]$-modules are just k-categories with G-action. So we know that the equivalence class of a 2-vector space $\mathcal{X}$ with G-action bijectively corresponds to the isomorphism class of a pair $(X, [v])$ of a finite G-set X and a cohomology class $[v]$ in $H^2(G, \mathrm{Map}(X, k^\times))$.

The category $\mathcal{V}[X, v]^G$ can be described as follows. An object of $\mathcal{V}[X, v]^G$ is a pair (V, f), where V is a family of vector spaces V_x for $x \in X$ and f is a family of linear maps $f(\sigma; x): V_x \to V_{\sigma x}$ for $\sigma \in G$, $x \in X$ satisfying

$$f(\sigma\tau; x) = f(\sigma; \tau x) \circ f(\tau; x)v(\sigma, \tau; x)$$

for all $\sigma, \tau \in G$, $x \in X$.

Suppose X is a transitive G-set and let K be the stabilizer of an element $x_0 \in X$. The map $v_0: K^2 \to k^\times$ defined by $v_0(\sigma, \tau) = v(\sigma, \tau; x_0)$ is a 2-cocycle on K. And we have Shapiro's isomorphism $H^2(G, \mathrm{Map}(X, k^\times)) \cong H^2(K, k^\times)$ in which $[v]$ corresponds to $[v_0]$. The pair (V, f) above is determined by the pair (V_{x_0}, f_0), where $f_0: K \to \mathrm{End}\, V_{x_0}$ is defined by $f_0(\sigma) = f(\sigma; x_0)$. Such a pair (V_{x_0}, f_0) is just a module over the skew group algebra $k[K, v_0]$ relative to the 2-cocycle v_0. Thus $\mathcal{V}[X, v]^G$ is equivalent to the category of $k[K, v_0]$-modules. Also $\mathcal{V}^G$ is the category of $k[G]$-modules. The action of $\mathcal{V}^G$ on $\mathcal{V}[X, v]^G$ is given by the tensor product through the restriction to the subgroup K.

6. Group actions on group tensor categories

In this section we apply the 2-equivalence of Section 4 to a group tensor category with G-action.

Any G-action on a group tensor category is obtained in the following way. Let A be a group with G-action denoted by $(\sigma, a) \mapsto {}^{\sigma}a$. Let

$$t \colon A \times A \times A \to k^{\times}$$
$$u \colon G \times A \times A \to k^{\times}$$
$$v \colon G \times G \times A \to k^{\times}$$

be maps satisfying

$$1 = \frac{t(b,c,d)t(a,bc,d)t(a,b,c)}{t(ab,c,d)t(a,b,cd)}$$

$$\frac{t(a,b,c)}{t({}^{\sigma}a,{}^{\sigma}b,{}^{\sigma}c)} = \frac{u(\sigma;b,c)u(\sigma;a,bc)}{u(\sigma;ab,c)u(\sigma;a,b)}$$

$$\frac{u(\sigma;{}^{\tau}a,{}^{\tau}b)u(\tau;a,b)}{u(\sigma\tau;a,b)} = \frac{v(\sigma,\tau;ab)}{v(\sigma,\tau;a)v(\sigma,\tau;b)}$$

$$\frac{v(\sigma\tau,\rho;a)v(\sigma,\tau;{}^{\rho}a)}{v(\tau,\rho;a)v(\sigma,\tau\rho;a)} = 1$$

for all $\sigma, \tau, \rho \in G$, $a, b, c, d \in A$. The first equation says t is a 3-cocycle of A, so we have the group tensor category $\mathcal{V}[A,t]$ of Section 5. A G-action on this tensor category is defined by

$$\sigma_*(a) = {}^{\sigma}a$$
$$\phi(\sigma,\tau)_a = v(\sigma,\tau;a)1_{\sigma\tau a}$$
$$\nu_a = \frac{1}{v(1,1;a)}1_a$$
$$\psi(\sigma)_{a,b} = u(\sigma;a,b)1_{\sigma(ab)}$$
$$\iota(\sigma) = \frac{1}{u(\sigma;1,1)}1_1$$

for $\sigma, \tau \in G$, $a, b \in A$.

By the definition of $C[G]$ in Section 2, we have $\mathcal{V}[A,t][G] = \mathcal{V}[A \rtimes G, s]$, where s is a 3-cocycle on the semi-direct product $A \rtimes G$ given by

$$s((a,\sigma),(b,\tau),(c,\rho)) = t(a,{}^{\sigma}b,{}^{\sigma\tau}c)u(\sigma;b,{}^{\tau}c)v(\sigma,\tau;c).$$

Our theorem applied to the G-tensor category $\mathcal{V}[A,t]$ says that the 2-functor

$$\mathcal{V}[A,t]^G\text{-Modk} \underset{(-)^G}{\longleftarrow} \mathcal{V}[A \rtimes G, s]\text{-Modk}$$

is a 2-equivalence. Assume k is algebraically closed. The property of being a 2-vector space is preserved under the above 2-equivalence. We saw in Section 5 that any $\mathcal{V}[A \rtimes G, s]$-module which is a 2-vector space is of the form $\mathcal{V}[X,r]$ for a finite $A \rtimes G$-set X and a map $r \colon (A \rtimes G)^2 \times X \to k^{\times}$ satisfying $i_*(s) = \partial r^{-1}$. Hence any $\mathcal{V}[A,t]^G$-module which is a 2-vector space is of the form $\mathcal{V}[X,r]^G$.

As an application of this, we can show

PROPOSITION. *If $|A|$ and $|G|$ are coprime and t is not a coboundary, then there exists no tensor functor $\mathcal{V}[A,t]^G \to \mathcal{V}$.*

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7. Generalization to $\mathcal{C}[G, w]$

In this section we generalize the 2-equivalence for $\mathcal{C}[G]$ to a 2-equivalence for a 3-cocycle deformation $\mathcal{C}[G, w]$.

We say that a tensor category $\mathcal{A}$ has a G-grading when $\mathcal{A}$ has a decomposition $\mathcal{A} = \bigoplus_{\sigma \in G} \mathcal{A}_\sigma$ as a k-category such that (i) if $A \in \mathcal{A}_\sigma, B \in \mathcal{A}_\tau$, then $A.B \in \mathcal{A}_{\sigma\tau}$, (ii) $I \in \mathcal{A}_1$. If $\mathcal{A}$ has a G-grading, a 3-cocycle w on G gives rise to a tensor category $\mathcal{A}^w$ as follows. The underlying k-category, tensor product and unit object of $\mathcal{A}^w$ are the same as those of $\mathcal{A}$, but the associativity and unit isomorphisms of $\mathcal{A}^w$ are given by

$$\alpha_{A,B,C}^{\mathcal{A}^w} = w(\sigma, \tau, \rho)\alpha_{A,B,C}^{\mathcal{A}}$$
$$\lambda_A^{\mathcal{A}^w} = w(1, 1, \sigma)^{-1}\lambda_A^{\mathcal{A}}$$
$$\rho_A^{\mathcal{A}^w} = w(\sigma, 1, 1)\rho_A^{\mathcal{A}}$$

for $A \in \mathcal{A}_\sigma$, $B \in \mathcal{A}_\tau$, $C \in \mathcal{A}_\rho$

Let $\mathcal{C}$ be a G-tensor category. Then the tensor category $\mathcal{C}[G]$ has the obvious G-grading. Hence the 3-cocycle w on G yields the tensor category $\mathcal{C}[G]^w$ which we denote by $\mathcal{C}[G, w]$.

THEOREM. *Let $\mathcal{M}$ be a $\mathcal{C}[G, w]$-module with underlying $\mathcal{C}$-module equivalent to $\mathcal{C}^n$ for $n > 0$. Then the 2-functors*

$$(\mathrm{End}_{\mathcal{C}[G,w]} \mathcal{M})^{\mathrm{op}}\text{-Mod}k \quad \underset{\mathrm{Hom}_{\mathcal{C}[G,w]}(\mathcal{M}, -)}{\overset{\mathcal{M} \otimes_{(\mathrm{End}_{\mathcal{C}[G,w]} \mathcal{M})^{\mathrm{op}}} -}{\rightleftarrows}} \quad \mathcal{C}[G, w]\text{-Mod}k$$

are quasi-inverse to each other.

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CHAPTER III

CATEGORICAL DEFORMATIONS OF ONE-DIMENSIONAL AFFINE TRANSFORMATION GROUPS

1. Summary

Let k be the complex field and G a finite group. We have the group algebra $k[G]$ and the function algebra $k(G) := \operatorname{Map}(G, k)$. Denote the category of $k[G]$-modules by $\operatorname{Rep}(G)$ and the category of $k(G)$-modules by $\operatorname{Vect}[G]$. We are concerned about deformations of $\operatorname{Rep}(G)$ and $\operatorname{Vect}[G]$ as tensor categories.

Look at $\operatorname{Vect}[G]$ first. A $k(G)$-module is a G-graded vector space $V = \bigoplus_{\sigma \in G} V_\sigma$, and the tensor product $W = U \otimes V$ of two modules U and V is graded as

$$W_\sigma = \bigoplus_{\sigma = \tau\rho} U_\tau \otimes V_\rho.$$

Simple modules are one-dimensional. They are labeled as $[\sigma]$ for $\sigma \in G$ so that

$$[\sigma]_\tau = \begin{cases} k & \text{if } \sigma = \tau, \\ 0 & \text{if } \sigma \neq \tau. \end{cases}$$

Then $[\sigma] \otimes [\tau] = [\sigma\tau]$.

If $\alpha \colon G \times G \times G \to k^\times$ is a 3-cocycle, $\operatorname{Vect}[G]$ is deformed to a tensor category $\operatorname{Vect}[G, \alpha]$. This has the same objects, morphisms, and tensor products as $\operatorname{Vect}[G]$. The only difference is in the associativity isomorphisms $(X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, which are a part of the structure of a tensor category. In $\operatorname{Vect}[G]$, the associativity $([\sigma] \otimes [\tau]) \otimes [\rho] \to [\sigma] \otimes ([\tau] \otimes [\rho])$ is the identity map on $[\sigma\tau\rho]$, while in $\operatorname{Vect}[G, \alpha]$ it is multiplication by the scalar $\alpha(\sigma, \tau, \rho)$. The pentagon axiom for associativity isomorphisms amounts to the cocycle condition for α.

Conversely, any tensor category with the same underlying category and the same tensor product operation as $\operatorname{Vect}[G]$ is of the form $\operatorname{Vect}[G, \alpha]$. Thus deformations of $\operatorname{Vect}[G]$ in such a sense are classified by the group $H^3(G, k^\times)$.

In contrast with this, any general procedure of deforming $\operatorname{Rep}(G)$ does not seem to be known. We will give some examples of deformations for small groups.

Central extensions. If K is a central subgroup of G, the set of irreducible characters of G is partitioned according to their restrictions to K. So the category $\mathcal{C} = \operatorname{Rep}(G)$ has a decomposition

$$\mathcal{C} = \bigoplus_{\lambda \in \hat{K}} \mathcal{C}_\lambda,$$

where $\hat{K} = \operatorname{Hom}(K, k^\times)$ and for $\lambda \in \hat{K}$, $\mathcal{C}_\lambda$ is the category of G-modules on which K acts through λ. If $X \in \mathcal{C}_\lambda$ and $Y \in \mathcal{C}_\mu$, then $X \otimes Y \in \mathcal{C}_{\lambda\mu}$. Thus we may say $\mathcal{C}$ has a $\hat{K}$-grading.

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If $\alpha \colon \hat{K}^3 \to k^\times$ is a 3-cocycle, $\mathcal{C}$ is deformed to a tensor category $\mathcal{C}^\alpha$ in a similar manner to the case of $\mathrm{Vect}[G]$. Namely, we let the associativity isomorphism $(X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ in $\mathcal{C}^\alpha$ for $X \in \mathcal{C}_\lambda$, $Y \in \mathcal{C}_\mu$, $Z \in \mathcal{C}_\nu$ to be the scalar multiplication by $\alpha(\lambda, \mu, \nu)$.

EXAMPLE 1. Let $G = D_8$, the dihedral group of order 8, and $K = Z(G) = Z_2$. Then $H^3(\hat{K}, k^\times) \cong Z_2$. Take a non-coboundary 3-cocycle α of $\hat{K}$. Then it turns out that $\mathcal{C}^\alpha \cong \mathrm{Rep}(Q_8)$.

EXAMPLE 2. Let $G = SL(2, q)$ with q odd and $K = Z(G) = \{\pm 1\}$. Let α be as above. Then it can be shown that the twisted category $\mathcal{C}^\alpha$ is equivalent to the module category for a Hopf algebra different from group algebras.

Semi-direct products. Next we consider a situation in which a group G acts on a group L. Form the semi-direct product LG. Let ρ be a 3-cocycle of LG which restricts to a coboundary of G. Put $\theta = \rho | L$. We have the category $\mathrm{Vect}[L, \theta]$ and ρ gives rise to an action of G on $\mathrm{Vect}[L, \theta]$ (Chapter II, Section 7). Then we have the tensor category $\mathrm{Vect}[L, \theta]^G$ of G-invariant objects in $\mathrm{Vect}[L, \theta]$. If L is abelian and $|L|$, $|G|$ are coprime, $\mathrm{Vect}[L, \theta]^G$ is a deformation of $\mathrm{Rep}(\hat{L}G)$.

EXAMPLE 3. Let $L = Z_3$, $G = Z_2$ and $LG \cong S_3$. We have $\mathrm{Ker}(H^3(LG) \to H^3(G)) \cong H^3(L)^G \cong Z_3$. Correspondingly three deformations of $\mathrm{Rep}(S_3)$ (including itself) are obtained. The two nontrivial ones are not representable as module categories over Hopf algebras. Moreover these are the only deformations of $\mathrm{Rep}(S_3)$.

EXAMPLE 4. Let $L = Z_2 \times Z_2$, $G = Z_3$ and $LG \cong A_4$. Then $\mathrm{Ker}(H^3(LG) \to H^3(G)) \cong Z_2$. We have one nontrivial deformation of $\mathrm{Rep}(A_4)$. This does not come from a Hopf algebra and is the unique nontrivial deformation.

Extraspecial 2-groups. An extraspecial 2-group has a unique irreducible non-linear character m. Let A be the group of linear characters. Then

$$m^2 \cong \sum_{a \in A} a.$$

Semi-simple tensor categories with fusion rule of this type were classified in [TY]. They are parameterized by pairs of nondegenerate symmetric bicharacter $A \times A \to k^\times$ and signs $\pm$. The signs correspond to the two types of extraspecial 2-groups.

One-dimensional affine transformation groups. The group $\mathbb{F}_q \rtimes \mathbb{F}_q^\times$ also has a unique non-linear character m and

$$m^2 = (q - 2)m + \sum_{a \in A} a$$

with $A = \mathrm{Hom}(\mathbb{F}_q^\times, k^\times)$. With a slight generalization we pose the problem: Classify semi-simple tensor categories of which the set of simple objects is a disjoint union $A \cup \{m\}$ of a group A and a one-point set $\{m\}$, and the fusion rule is

$$a \otimes b \cong ab,$$
$$a \otimes m \cong m, \quad m \otimes a \cong m$$
$$m \otimes m \cong \underbrace{m \oplus \cdots \oplus m}_{N} \oplus \bigoplus_{a \in A} a$$

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for $a, b \in A$ with $N \in \mathbb{N}$.

At present we have a few results for small values of N.

- If $N = 1$, there are just three such categories. They are $\mathrm{Rep}(\mathbb{F}_3 \rtimes \mathbb{F}_3^\times) = \mathrm{Rep}(S_3)$ and its twists in Example 3.
- If $N = 2$, there are just two such categories. They are $\mathrm{Rep}(\mathbb{F}_4 \rtimes \mathbb{F}_4^\times) = \mathrm{Rep}(A_4)$ and its twist in Example 4.
- If $N = 6$, there is such a category other than $\mathrm{Rep}(\mathbb{F}_8 \rtimes \mathbb{F}_8^\times)$.

In this chapter we outline our attempt to solve the problem.

2. Structure constants

Our aim is to classify semi-simple tensor category having the set $A \cup \{m\}$ of simple objects, with A a finite group, and fusion rule

$$a \otimes b \cong ab$$
$$a \otimes m \cong m, \quad m \otimes a \cong m$$
$$m \otimes m \cong Vm \oplus \bigoplus_{a \in A} a$$

for $a, b \in A$, where V is a vector space. Here Vm means the direct sum of $\dim V$ copy of m. (In general for a vector space U and an object x of a k-linear category $\mathcal{C}$, an object Ux of $\mathcal{C}$ is defined and it behaves naturally in U and x.)

Choose isomorphisms of the above fusion rule and name them and their components as

$$[a, b] : a \otimes b \to ab$$
$$[a, m] : a \otimes m \to m$$
$$[m, a] : m \otimes a \to m$$
$$[m, m, m] : m \otimes m \to Vm$$
$$[m, m, a] : m \otimes m \to a.$$

We use the following notation for monoidal structures:

$$\mathbf{a}_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z) \quad \text{associativity isomorphism}$$
$$\mathbf{l}_x : x \otimes I \to x \quad \text{left unit isomorphism}$$
$$\mathbf{r}_x : I \otimes x \to x \quad \text{right unit isomorphism.}$$

We describe the associativity $\mathbf{a}$ in terms of scalars and linear maps.

- (a, b, c): For $a, b, c \in A$, consider the composites

$$[ab, c] \circ ([a, b] \cdot c) : (a \cdot b) \cdot c \to ab \cdot c \to abc$$
$$[a, bc] \circ (a \cdot [b, c]) : a \cdot (b \cdot c) \to a \cdot bc \to abc.$$

The isomorphism $\mathbf{a}_{a,b,c} : (a \cdot b) \cdot c \to a \cdot (b \cdot c)$ determines a nonzero scalar $\alpha(a, b, c) \in k$ so that

$$[a, bc] \circ (a \cdot [b, c]) = \alpha(a, b, c) 1_{abc} \circ [ab, c] \circ ([a, b] \cdot c).$$

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For brevity we write this situation as

$$(a \cdot b) \cdot c \to ab \cdot c \to abc \qquad p_l$$
$$a \cdot (b \cdot c) \to a \cdot bc \to abc \qquad p_r$$

$$p_r \mathbf{a} = \alpha(a,b,c) p_l.$$

- (a,b,m):

$$(a \cdot b) \cdot m \to ab \cdot m \to m \qquad p_l$$
$$a \cdot (b \cdot m) \to a \cdot m \to m \qquad p_r$$

$$p_r \mathbf{a} = \alpha_3(a,b) p_l$$

with $\alpha_3(a,b) \in k$.

- (a,m,b):

$$(a \cdot m) \cdot b \to m \cdot b \to m \qquad p_l$$
$$a \cdot (m \cdot b) \to a \cdot m \to m \qquad p_r$$

$$p_r \mathbf{a} = \alpha_2(a,b) p_l$$

with $\alpha_2(a,b) \in k$.

- (m,a,b):

$$(m \cdot a) \cdot b \to m \cdot b \to m \qquad p_l$$
$$m \cdot (a \cdot b) \to m \cdot ab \to m \qquad p_r$$

$$p_r \mathbf{a} = \alpha_1(a,b) p_l$$

with $\alpha_1(a,b) \in k$.

- (a,m,m): For $a,b \in A$, consider the composites

$$[m,m,m] \circ ([a,m] \cdot m) \colon (a \cdot m) \cdot m \to m \cdot m \to Vm$$
$$[m,m,b] \circ ([a,m] \cdot m) \colon (a \cdot m) \cdot m \to m \cdot m \to b$$
$$V[a,m] \circ (a \cdot [m,m,m]) \colon a \cdot (m \cdot m) \to Va \cdot m \to Vm$$
$$[a,b] \circ (a \cdot [m,m,b]) \colon a \cdot (m \cdot m) \to a \cdot b \to ab.$$

The isomorphism $\mathbf{a}_{a,m,m}$ determines a linear isomorphism $\beta_1(a,m) \colon V \to V$ and a nonzero scalar $\beta_1(a,ab)$ so that

$$V[a,m] \circ (a \cdot [m,m,m] \circ \mathbf{a}_{a,m,m} = \beta_1(a,m)m \circ [m,m,m] \circ ([a,m] \cdot m)$$
$$[a,b] \circ (a \cdot [m,m,b]) \circ \mathbf{a}_{a,m,m} = \beta_1(a,ab)1_{ab} \circ [m,m,ab] \circ ([a,m] \cdot m).$$

We write this situation as

$$(a \cdot m) \cdot m \to m \cdot m \to Vm \qquad p_l(m,m)$$
$$\searrow$$
$$b \qquad p_l(m,b)$$

$$a \cdot (m \cdot m) \to Va \cdot m \to Vm \qquad p_r(m,m)$$
$$\searrow$$
$$a \cdot b \to ab \qquad p_r(b,ab)$$

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$$p_r(m,m)\mathbf{a} = \beta_1(a,m)p_l(m,m)$$
$$p_r(b,ab)\mathbf{a} = \beta_1(a,ab)p_l(m,ab)$$

where

$$\beta_1(a,m)\colon V \to V$$
$$\beta_1(a,ab)\colon k \to k.$$

- (m,a,m):

$$(m \cdot a) \cdot m \to m \cdot m \to Vm \qquad p_l(m,m)$$
$$\searrow$$
$$b \qquad p_l(m,b)$$

$$m \cdot (a \cdot m) \to m \cdot m \to Vm \qquad p_r(m,m)$$
$$\searrow$$
$$b \qquad p_r(m,b)$$

$$p_r(m,m)\mathbf{a} = \beta_2(a,m)p_l(m,m)$$
$$p_r(m,b)\mathbf{a} = \beta_2(a,b)p_l(m,b)$$

where

$$\beta_2(a,m)\colon V \to V$$
$$\beta_2(a,b)\colon k \to k.$$

- (m,m,a):

$$(m \cdot m) \cdot a \to Vm \cdot a \to Vm \qquad p_l(m,m)$$
$$\searrow$$
$$b \cdot a \to ba \qquad p_l(b,ba)$$

$$m \cdot (m \cdot a) \to m \cdot m \to Vm \qquad p_r(m,m)$$
$$\searrow$$
$$b \qquad p_r(m,b)$$

$$p_r(m,m)\mathbf{a} = \beta_3(a,m)p_l(m,m)$$
$$p_r(m,ba)\mathbf{a} = \beta_3(a,ba)p_l(b,ba)$$

where

$$\beta_3(a,m)\colon V \to V$$
$$\beta_3(a,ba)\colon k \to k.$$

- (m,m,m):

$$(m \cdot m) \cdot m \to Vm \cdot m \to VVm \qquad p_l(m,m)$$
$$\searrow$$
$$Va \qquad p_l(m,a)$$

$$b \cdot m \to m \qquad p_l(b,m)$$

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$$m \cdot (m \cdot m) \to V m \cdot m \to V V m \qquad p_r(m,m)$$
$$\searrow \qquad \qquad \searrow \qquad \qquad V a \qquad p_r(m,a)$$
$$m \cdot b \quad \to \quad m \qquad p_r(b,m)$$

$$p_r(m,m)\mathbf{a} = \gamma(m,m)p_l(m,m) + \sum_{b'} \gamma(m,b')p_l(b',m)$$
$$p_r(m,a)\mathbf{a} = \gamma(a)p_l(m,a)$$
$$p_r(b,m)\mathbf{a} = \gamma(b,m)p_l(m,m) + \sum_{b'} \gamma(b,b')p_l(b',m)$$

where

$$\gamma(m,m)\colon VV \to VV$$
$$\gamma(m,b')\colon k \to VV$$
$$\gamma(b,m)\colon VV \to k$$
$$\gamma(b,b')\colon k \to k$$
$$\gamma(a)\colon V \to V.$$

In summary, the associativity isomorphisms are specified by the following data:

$$\alpha(a,b,c) \in k$$
$$\alpha_1(a,b), \alpha_2(a,b), \alpha_3(a,b) \in k$$
$$\beta_1(a,b), \beta_2(a,b), \beta_3(a,b) \in k$$
$$\beta_1(a,m), \beta_2(a,m), \beta_3(a,m)\colon V \to V$$
$$\gamma(m,m)\colon VV \to VV$$
$$\gamma(m,b')\colon k \to VV$$
$$\gamma(b,m)\colon VV \to k$$
$$\gamma(b,b') \in k$$
$$\gamma(a)\colon V \to V.$$

3. Triangle equations

The unit object is $1 \in A$. Choose $[1,a], [a,1], [1,m], [m,1]$ so that

$$[1,a] = \mathbf{l}_a\colon 1 \otimes a \to a, \qquad [a,1] = \mathbf{r}_a\colon a \otimes 1 \to a,$$
$$[1,m] = \mathbf{l}_m\colon 1 \otimes m \to m, \quad [m,1] = \mathbf{r}_m\colon m \otimes 1 \to m.$$

Then the triangle equations

$$\mathbf{a}_{X,I,Y}(X \otimes \mathbf{l}_Y) = \mathbf{r}_X \otimes Y$$

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for $(a,1,c), (m,1,b), (a,1,m), (m,1,m)$ yield

$$\alpha(a,1,c) = 1,$$
$$\alpha_1(1,b) = 1,$$
$$\alpha_3(a,1) = 1,$$
$$\beta_2(1,m) = 1_V, \quad \beta_2(1,b) = 1.$$

The triangle equations

$$(X \otimes 1_Y) \circ \mathbf{a}_{X,Y,I} = 1_{X \otimes Y}$$
$$\mathbf{r}_{X \otimes Y} \circ \mathbf{a}_{I,X,Y} = \mathbf{r}_X \otimes Y$$

for $(1,b,c), (a,b,1), (m,a,1), (a,m,1), (1,m,b), (1,b,m), (1,m,m), (m,m,1)$ yield

$$\alpha(1,b,c) = 1,$$
$$\alpha(a,b,1) = 1,$$
$$\alpha_1(a,1) = 1,$$
$$\alpha_2(a,1) = 1,$$
$$\alpha_2(1,b) = 1,$$
$$\alpha_3(1,b) = 1,$$
$$\beta_1(1,m) = 1_V, \quad \beta_1(1,b) = 1,$$
$$\beta_3(1,m) = 1_V, \quad \beta_3(1,b) = 1.$$

4. Change of bases

We next examine how the structural constants depend on the choice of isomorphisms in the fusion rule. Let

$$[a,b]' : a \otimes b \to ab$$
$$[a,m]' : a \otimes m \to m$$
$$[m,a]' : m \otimes a \to m$$
$$[m,m,m]' : m \otimes m \to Vm$$
$$[m,m,a]' : m \otimes m \to a$$

be another choice with the normalization condition of Section 3. Then there exist

$$\theta(a,b), \theta_1(a), \theta_2(a), \phi(a) \in k^\times$$
$$\phi(m) \in GL(V)$$

such that

$$[a,b]' = \theta(a,b)[a,b]$$
$$[a,m]' = \theta_2(a)[a,m]$$
$$[m,a]' = \theta_1(a)[m,a]$$
$$[m,m,m]' = \phi(m)[m,m,m]$$
$$[m,m,a]' = \phi(a)[m,m,a]$$

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The normalization conditions for the both choices imply

$$\theta(1,b) = \theta(a,1) = 1, \quad \theta_1(1) = 1, \quad \theta_2(1) = 1.$$

The new choice will yield new structural constants $\alpha'(a,b,c)$, $\alpha_1'(a,b)$, ... , which are related to old ones as follows.

- (a,b,c)
$$\alpha'(a,b,c)\theta(a,b)\theta(ab,c) = \theta(b,c)\theta(a,bc)\alpha(a,b,c)$$

- (a,b,m)
$$\alpha_3'(a,b)\theta(a,b)\theta_2(ab) = \theta_2(b)\theta_2(a)\alpha_3(a,b)$$

- (a,m,b)
$$\alpha_2'(a,b)\theta_2(a)\theta_1(b) = \theta_1(b)\theta_2(a)\alpha_2(a,b)$$

- (m,a,b)
$$\alpha_1'(a,b)\theta_1(a)\theta_1(b) = \theta(a,b)\theta_1(ab)\alpha_1(a,b)$$

- (a,m,m)
$$\beta_1'(a,m) \circ \theta_2(a)\phi(m) = \phi(m)\theta_2(a) \circ \beta_1(a,m)$$
$$\beta_1'(a,ab)\theta_2(a)\phi(ab) = \phi(b)\theta(a,b)\beta_1(a,ab)$$

- (m,a,m)
$$\beta_2'(a,m) \circ \theta_1(a)\phi(m) = \theta_2(a)\phi(m) \circ \beta_2(a,m)$$
$$\beta_2'(a,b)\theta_1(a)\phi(b) = \theta_2(a)\phi(b)\beta_2(a,b)$$

- (m,m,a)
$$\beta_3'(a,m) \circ \phi(m)\theta_1(a) = \theta_1(a)\phi(m) \circ \beta_3(a,m)$$
$$\beta_3'(a,ba)\phi(b)\theta(b,a) = \theta_1(a)\phi(ba)\beta_3(a,ba)$$

- (m,m,m)
$$\gamma'(m,m) \circ \phi(m)\phi(m) = \phi(m)\phi(m) \circ \gamma(m,m)$$
$$\gamma'(m,b') \circ \phi(b')\theta_2(b') = \phi(m)\phi(m) \circ \gamma(m,b')$$
$$\gamma'(b,m) \circ \phi(m)\phi(m) = \phi(b)\theta_1(b) \circ \gamma(b,m)$$
$$\gamma'(b,b') \circ \phi(b')\theta_2(b') = \phi(b)\theta_1(b) \circ \gamma(b,b')$$
$$\gamma'(a) \circ \phi(m)\phi(a) = \phi(m)\phi(a) \circ \gamma(a)$$

5. Structure constants for the one-dimensional affine groups

Let $F = \mathbb{F}_q$, $F^\times = F - \{0\}$. Let G be the semi-direct product $F \rtimes F^\times$. Namely $G = \{(a,b) \mid a \in F, b \in F^\times\}$ with multiplication

$$(a,b)(a',b') = (a + ba', bb').$$

The simple G-modules are named as L_λ for $\lambda \in \widehat{F^\times}$ and M. The module L_λ is one dimensional with

$$\text{basis:} \quad \langle \lambda \rangle$$
$$\text{action:} \quad (a,b)\langle \lambda \rangle = \lambda(b)\langle \lambda \rangle.$$

Fix $1 \neq \chi_1 \in \hat{F}$. The module M is $q-1$ dimensional with

$$\text{basis} \quad [a] \text{ for } a \in F^\times$$
$$\text{action} \quad (a,1)[c] = \chi_1(ac)[c],$$
$$(0,b)[c] = [b^{-1}c].$$

Let V be the vector space with basis (x) for $x \in F - \{0,1\}$. We have G-maps

$$L_\lambda \otimes L_\mu \to L_{\lambda\mu}$$
$$\langle \lambda \rangle \otimes \langle \mu \rangle \mapsto \langle \lambda\mu \rangle$$

$$L_\lambda \otimes M \to M$$
$$\langle \lambda \rangle \otimes [a] \mapsto \lambda(a)[a]$$

$$M \otimes M \to V \otimes M$$
$$[a] \otimes [b] \mapsto \begin{cases} (-\frac{b}{a}) \otimes [a+b] & \text{if } a+b \neq 0 \\ 0 & \text{if } a+b = 0 \end{cases}$$

$$M \otimes M \to L_\lambda$$
$$[a] \otimes [b] \mapsto \delta_{a+b,0}\lambda(a)^{-1}\langle \lambda \rangle.$$

With this choice of maps, the structure constants of Section 2 are given as follows.

$$\alpha(L_\lambda, L_\mu, L_\nu) = 1,$$
$$\alpha_1(L_\lambda, L_\mu) = 1, \alpha_2(L_\lambda, L_\mu) = 1, \alpha_3(L_\lambda, L_\mu) = 1$$
$$\beta_1(L_\lambda, L_\mu) = 1, \beta_2(L_\lambda, L_\mu) = 1, \beta_3(L_\lambda, L_\mu) = 1$$

$$V \to V$$
$$\beta_1(L_\lambda, M) \colon (x) \mapsto \lambda(1-x)(x)$$
$$\beta_2(L_\lambda, M) \colon (x) \mapsto \lambda(x)(x)$$
$$\beta_3(L_\lambda, M) \colon (x) \mapsto \lambda(\frac{x}{x-1})(x)$$
$$\gamma(L_\lambda) \colon (x) \mapsto \lambda(1-x)(1-\frac{1}{x})$$

$$\gamma(M, M)\colon V \otimes V \to V \otimes V$$

$$(x) \otimes (y) \mapsto \begin{cases} ((1 - \tfrac{1}{x})y) \otimes (x + y - xy) & \text{if } \tfrac{1}{x} + \tfrac{1}{y} \neq 1 \\ 0 & \text{if } \tfrac{1}{x} + \tfrac{1}{y} = 1 \end{cases}$$

$$\gamma(L_\mu, M)\colon V \otimes V \to k$$

$$(x) \otimes (y) \mapsto \begin{cases} \mu(x^{-1}) & \text{if } \tfrac{1}{x} + \tfrac{1}{y} = 1 \\ 0 & \text{if } \tfrac{1}{x} + \tfrac{1}{y} \neq 1 \end{cases}$$

$$\gamma(M, L_\lambda)\colon k \to V \otimes V$$

$$1 \mapsto \frac{1}{q - 1} \sum_{u \neq 0, 1} \lambda(u)^{-1}(u) \otimes (1 - u)$$

$$\gamma(L_\lambda, L_\mu)\colon k \to k$$

$$1 \mapsto \frac{1}{q - 1}$$

6. Writing down pentagon equations

We now return to the general case. The pentagon equation

$$(\mathbf{a}_{X,Y,Z} \otimes W) \circ \mathbf{a}_{X,Y \otimes Z,W} \circ (X \otimes \mathbf{a}_{Y,Z,W}) = \mathbf{a}_{X \otimes Y,Z,W} \circ \mathbf{a}_{X,Y,Z \otimes W}$$

for each quadruple (X, Y, Z, W) of simple objects is expressed in terms of the structural constants as follows.

- (a, b, c, d)
$$\alpha(b, c, d)\alpha(a, bc, d)\alpha(a, b, c) = \alpha(a, b, cd)\alpha(ab, c, d)$$

- (a, b, c, m)
$$\alpha_3(b, c)\alpha_3(a, bc)\alpha(a, b, c) = \alpha_3(a, b)\alpha_3(ab, c)$$

- (a, b, m, c)
$$\alpha_2(b, c)\alpha_2(a, c)\alpha_3(a, b) = \alpha_3(a, b)\alpha_2(ab, c)$$

- (a, m, b, c)
$$\alpha_1(b, c)\alpha_2(a, c)\alpha_2(a, b) = \alpha_2(a, bc)\alpha_1(b, c)$$

- (m, a, b, c)
$$\alpha(a, b, c)\alpha_1(ab, c)\alpha_1(a, b) = \alpha_1(a, bc)\alpha_1(b, c)$$

- (a, b, m, m)
$$\beta_1(b, m) \circ \beta_1(a, m) \circ \alpha_3(a, b)V = V\alpha_3(a, b) \circ \beta_1(ab, m)$$
$$\beta_1(b, bc) \circ \beta_1(a, abc) \circ \alpha_3(a, b) = \alpha(a, b, c) \circ \beta_1(ab, abc)$$

- (a, m, b, m)
$$\beta_2(b, m) \circ \beta_1(a, m) \circ \alpha_2(a, b)V = \beta_1(a, m) \circ \beta_2(b, m)$$
$$\beta_2(b, c) \circ \beta_1(a, ac) \circ \alpha_2(a, b) = \beta_1(a, ac) \circ \beta_2(b, ac)$$

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- (a, m, m, b)

$$\beta_3(b, m) \circ V\alpha_2(a, b) \circ \beta_1(a, m) = \beta_1(a, m) \circ \beta_3(b, m)$$
$$\beta_3(b, cb) \circ \alpha(a, c, b) \circ \beta_1(a, ac) = \beta_1(a, acb) \circ \beta_3(b, acb)$$

- (m, m, a, b)

$$\alpha_1(a, b)V \circ \beta_3(b, m) \circ \beta_3(a, m) = \beta_3(ab, m) \circ V\alpha_1(a, b)$$
$$\alpha_1(a, b) \circ \beta_3(b, cab) \circ \beta_3(a, ca) = \beta_3(ab, cab) \circ \alpha(c, a, b)$$

- (m, a, m, b)

$$\alpha_2(a, b)V \circ \beta_3(b, m) \circ \beta_2(a, m) = \beta_2(a, m) \circ \beta_3(b, m)$$
$$\alpha_2(a, b) \circ \beta_3(b, cb) \circ \beta_2(a, c) = \beta_2(a, cb) \circ \beta_3(b, cb)$$

- (m, a, b, m)

$$\alpha_3(a, b)V \circ \beta_2(ab, m) \circ \alpha_1(a, b)V = \beta_2(a, m) \circ \beta_2(b, m)$$
$$\alpha_3(a, b) \circ \beta_2(ab, c) \circ \alpha_1(a, b) = \beta_2(a, c) \circ \beta_2(b, c)$$

- (a, m, m, m)

$$\gamma(m, m) \circ V\beta_1(a, m) \circ \beta_1(a, m)V = V\beta_1(a, m) \circ \gamma(m, m)$$
$$\gamma(m, c') \circ \alpha_3(a, c') \circ \beta_1(a, ac') = V\beta_1(a, m) \circ \gamma(m, ac')$$
$$\gamma(b) \circ V\beta_1(a, ab) \circ \beta_1(a, m) = V\beta_1(a, ab) \circ \gamma(ab)$$
$$\gamma(c, m) \circ V\beta_1(a, m) \circ \beta_1(a, m)V = \alpha_2(a, c) \circ \gamma(c, m)$$
$$\gamma(c, c') \circ \alpha_3(a, c') \circ \beta_1(a, ac') = \alpha_2(a, c) \circ \gamma(c, ac')$$

- (m, a, m, m)

$$\beta_1(a, m)V \circ \gamma(m, m) \circ \beta_2(a, m)V = V\beta_2(a, m) \circ \gamma(m, m)$$
$$\beta_1(a, m)V \circ \gamma(m, c') \circ \beta_2(a, c') = V\beta_2(a, m) \circ \gamma(m, c')$$
$$\beta_1(a, m) \circ \gamma(b) \circ \beta_2(a, m) = V\beta_2(a, b) \circ \gamma(b)$$
$$\beta_1(a, ac) \circ \gamma(ac, m) \circ \beta_2(a, m)V = \alpha_1(a, c) \circ \gamma(c, m)$$
$$\beta_1(a, ac) \circ \gamma(ac, c') \circ \beta_2(a, c') = \alpha_1(a, c) \circ \gamma(c, c')$$

- (m, m, a, m)

$$\beta_2(a, m)V \circ \gamma(m, m) \circ \beta_3(a, m)V = \gamma(m, m) \circ V\beta_2(a, m)$$
$$\beta_2(a, m)V \circ \gamma(m, c'a) \circ \beta_3(a, c'a) = \gamma(m, c') \circ \alpha_3(c', a)$$
$$\beta_2(a, m) \circ \gamma(b) \circ \beta_3(a, m) = \gamma(b) \circ V\beta_2(a, b)$$
$$\beta_2(a, c) \circ \gamma(c, m) \circ \beta_3(a, m)V = \gamma(c, m) \circ V\beta_2(a, m)$$
$$\beta_2(a, c) \circ \gamma(c, c'a) \circ \beta_3(a, c'a) = \gamma(c, c') \circ \alpha_3(c', a)$$

- (m, m, m, a)

$$\beta_3(a, m)V \circ V\beta_3(a, m) \circ \gamma(m, m) = \gamma(m, m) \circ V\beta_3(a, m)$$

$$\beta_3(a, m)V \circ V\beta_3(a, m) \circ \gamma(m, c') = \gamma(m, c') \circ \alpha_2(c', a)$$

$$\beta_3(a, m) \circ V\beta_3(a, ba) \circ \gamma(b) = \gamma(ba) \circ V\beta_3(a, ba)$$

$$\beta_3(a, ca) \circ \alpha_1(c, a) \circ \gamma(c, m) = \gamma(ca, m) \circ V\beta_3(a, m)$$

$$\beta_3(a, ca) \circ \alpha_1(c, a) \circ \gamma(c, c') = \gamma(ca, c') \circ \alpha_2(c', a)$$

- (m, m, m, m)

$$\gamma(m, m)V \circ V\gamma(m, m) \circ \gamma(m, m)V + \sum_{c'} \gamma(m, c')V \circ \beta_2(c', m) \circ \gamma(c', m)V$$
$$= V\gamma(m, m) \circ TV \circ V\gamma(m, m)$$

$$\gamma(m, m)V \circ V\gamma(m, b') \circ \gamma(b') = V\gamma(m, m) \circ TV \circ V\gamma(m, b')$$

$$\gamma(m, m)V \circ V\gamma(m, m) \circ \gamma(m, c'')V + \sum_{c'} \gamma(m, c')V \circ \beta_2(c', m) \circ \gamma(c', c'')V$$
$$= V\gamma(m, c'') \circ \beta_1(c'', m)$$

$$\gamma(m, m) \circ V\gamma(a) \circ \gamma(m, m) + \sum_{c'} \gamma(m, c') \circ \beta_2(c', a) \circ \gamma(c', m)$$
$$= T \circ \gamma(a)\gamma(a)$$

$$\gamma(m, m) \circ V\gamma(a) \circ \gamma(m, c'') + \sum_{c'} \gamma(m, c') \circ \beta_2(c', a) \circ \gamma(c', c'') = 0$$

$$\gamma(b) \circ V\gamma(b, m) \circ \gamma(m, m)V = V\gamma(b, m) \circ TV \circ V\gamma(m, m)$$

$$\gamma(b) \circ V\gamma(b, m) \circ \gamma(m, c')V = V\gamma(b, c') \circ \beta_1(c', m)$$

$$\gamma(b) \circ V\gamma(b, b') \circ \gamma(b') = V\gamma(b, m) \circ TV \circ V\gamma(m, b')$$

$$\gamma(c, m)V \circ V\gamma(m, m) \circ \gamma(m, m)V + \sum_{c'} \gamma(c, c')V \circ \beta_2(c', m) \circ \gamma(c', m)V$$
$$= \beta_3(c, m) \circ V\gamma(c, m)$$

$$\gamma(c, m)V \circ V\gamma(m, b') \circ \gamma(b') = \beta_3(c, m) \circ V\gamma(c, b')$$

$$\gamma(c, m)V \circ V\gamma(m, m) \circ \gamma(m, c'')V + \sum_{c'} \gamma(c, c')V \circ \beta_2(c', m) \circ \gamma(c', c'')V = 0$$

$$\gamma(c, m) \circ V\gamma(d) \circ \gamma(m, m) + \sum_{c'} \gamma(c, c') \circ \beta_2(c', d) \circ \gamma(c', m) = 0$$

$$\gamma(c, m) \circ V\gamma(d) \circ \gamma(m, c'') + \sum_{c'} \gamma(c, c') \circ \beta_2(c', d) \circ \gamma(c', c'')$$
$$= \delta_{c'', dc^{-1}} \beta_3(c, dc^{-1}) \circ \beta_1(dc^{-1}, d)$$

7. Solving pentagon equations

1. First reduction. With the choice of $\theta(a,b), \theta_1(a), \theta_2(b), \phi(a) \in k^\times$ such that

$$\theta(a,b) = \alpha_3(a,b),$$
$$\theta_2(a) = 1,$$
$$\theta_1(a) = \beta_2(a,1)$$
$$\frac{\phi(a^{-1})}{\phi(1)} = \theta(a,a^{-1})^{-1}\beta_1(a,1)^{-1},$$

we have

$$\alpha_3'(a,b) = 1, \beta_2'(a,1) = 1, \beta_1'(a,1) = 1.$$

So we may assume

$$\alpha_3(a,b) = 1, \beta_2(a,1) = 1, \beta_1(a,1) = 1.$$

We assume furthermore $\gamma(1,1) \neq 0$. Then the equations of Section 6 reduce to the following.

$$\alpha(a,b,c) = 1$$
$$\alpha_1(a,b) = 1, \alpha_3(a,b) = 1$$
$$\alpha_2(ab,c) = \alpha_2(a,c)\alpha_2(b,c)$$
$$\alpha_2(b,a) = \alpha_2(a,b)$$
$$\beta_1(a,b) = 1, \beta_3(a,b) = 1$$
$$\beta_2(a,b) = \alpha_2(a,b)$$
$$\gamma(a,b) = \frac{\gamma(1,1)}{\alpha_2(a,b)}$$

$$\beta_1(b,m) \circ \beta_1(a,m) = \beta_1(ab,m)$$
$$\beta_2(a,m) \circ \beta_2(b,m) = \beta_2(ab,m)$$
$$\beta_3(b,m) \circ \beta_3(a,m) = \beta_3(ab,m)$$
$$\beta_1(a,m) \circ \beta_2(b,m) = \alpha_2(a,b)\beta_2(b,m) \circ \beta_1(a,m)$$
$$\beta_2(a,m) \circ \beta_3(b,m) = \alpha_2(a,b)\beta_3(b,m) \circ \beta_2(a,m)$$
$$\beta_1(a,m) \circ \beta_3(b,m) = \alpha_2(a,b)\beta_3(b,m) \circ \beta_1(a,m)$$

$$\gamma(a) = \gamma(1) \circ \beta_1(a,m)$$
$$= \beta_3(a,m) \circ \gamma(1)$$

$$\beta_1(a,m)^{-1} = \gamma(1) \circ \beta_2(a,m) \circ \gamma(1)^{-1}$$
$$\beta_3(a,m)^{-1} = \gamma(1)^{-1} \circ \beta_2(a,m) \circ \gamma(1)$$
$$\beta_1(a,m) = \gamma(1)^{-1} \circ \beta_3(a,m) \circ \gamma(1)$$

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$$\gamma(m, a) = \beta_2(a, m)^{-1} V \circ \gamma(m, 1)$$
$$= V\beta_1(a, m)^{-1} \circ \gamma(m, 1)$$
$$\gamma(a, m) = \gamma(1, m) \circ \beta_2(a, m)^{-1} V$$
$$= \gamma(1, m) \circ V\beta_3(a, m)^{-1}$$

$$\gamma(1, m) \circ \beta_1(a, m)\beta_1(a, m) = \gamma(1, m)$$
$$\gamma(1, m) \circ \beta_3(a, m)^{-1}\beta_2(a, m) = \gamma(1, m)$$
$$\gamma(1, m) \circ \beta_2(a, m)\beta_3(a, m)^{-1} = \gamma(1, m)$$
$$\beta_3(a, m)\beta_3(a, m) \circ \gamma(m, 1) = \gamma(m, 1)$$
$$\beta_1(a, m)^{-1}\beta_2(a, m) \circ \gamma(m, 1) = \gamma(m, 1)$$
$$\beta_2(a, m)\beta_1(a, m)^{-1} \circ \gamma(m, 1) = \gamma(m, 1)$$

$$\gamma(m, m) \circ \beta_1(a, m)\beta_1(a, m) = V\beta_1(a, m) \circ \gamma(m, m)$$
$$\beta_3(a, m)\beta_3(a, m) \circ \gamma(m, m) = \gamma(m, m) \circ V\beta_3(a, m)$$
$$\gamma(m, m) \circ \beta_2(a, m)V = \beta_1(a, m)^{-1}\beta_2(a, m) \circ \gamma(m, m)$$
$$\beta_2(a, m)V \circ \gamma(m, m) = \gamma(m, m) \circ \beta_3(a, m)^{-1}\beta_2(a, m)$$

$$\gamma(m, m)V \circ V\gamma(m, m) \circ \gamma(m, m)V$$
$$+ \sum_{c'} \left[\beta_2(c', m)^{-1} V \circ \gamma(m, 1) \circ \gamma(1, m) \circ \beta_2(c', m)^{-1}V \right] \beta_2(c', m)$$
$$= V\gamma(m, m) \circ TV \circ V\gamma(m, m)$$

$$\gamma(m, m)V \circ V\gamma(m, 1) \circ \gamma(1) = V\gamma(m, m) \circ TV \circ V\gamma(m, 1)$$
$$\gamma(1) \circ V\gamma(1, m) \circ \gamma(m, m)V = V\gamma(1, m) \circ TV \circ V\gamma(m, m)$$

$$\gamma(m, m)V \circ V\gamma(m, m) \circ \gamma(m, 1)V + \gamma(1, 1)\left[\sum_{c'} \beta_2(c', m)^{-1} V\beta_2(c', m) \right] \circ \gamma(m, 1)V$$
$$= V\gamma(m, 1)$$

$$\gamma(1, m)V \circ V\gamma(m, m) \circ \gamma(m, m)V + \gamma(1, 1)\gamma(1, m)V \circ \sum_{c'} \beta_2(c', m)^{-1} V\beta_2(c', m)$$
$$= V\gamma(1, m)$$

$$\gamma(m,m) \circ V\gamma(1) \circ \gamma(m,m) + \sum_{c'} \beta_2(c',m)^{-1} V \circ \gamma(m,1) \circ \gamma(1,m) \circ \beta_2(c',m)^{-1} V$$

$$= T \circ \gamma(1)\gamma(1)$$

$$\gamma(m,m) \circ V[\gamma(1) \circ \beta_1(a,m)] \circ \gamma(m,1)$$
$$+ \gamma(1,1)V\left[\sum_{c'} \beta_1(c',m)^{-1}\alpha_2(c',a)\right] \circ \gamma(m,1) = 0$$

$$\gamma(1,m) \circ V[\beta_3(d,m) \circ \gamma(1)] \circ \gamma(m,m)$$
$$+ \gamma(1,1)\gamma(1,m) \circ V\left[\sum_{c'} \alpha_2(c',d)\beta_3(c',m)^{-1}\right] = 0$$

$$V\gamma(1,m) \circ \gamma(m,1)V = \gamma(1,1)\gamma(1)^{-1}$$
$$\gamma(1,m)V \circ V\gamma(m,1) = \gamma(1,1)\gamma(1)^{-1}$$
$$V\gamma(1,m) \circ TV \circ V\gamma(m,1) = \gamma(1,1)\gamma(1)^2$$

$$\gamma(1,m)V \circ \beta_2(c,m)^{-1}\gamma(m,m) \circ \gamma(m,1)V$$
$$+ \gamma(1,1)^2 \sum_{c'} \alpha_2(c,c')^{-1}\beta_2(c',m) = 0$$

$$\gamma(1,m) \circ V[\beta_3(d,m) \circ \gamma(1)] \circ \gamma(m,1) + \gamma(1,1)^2 \sum_{c'} \alpha_2(c',d) = \delta_{d,1}$$

2. Possible change of bases. The base change given by

$$[a,b]' = \theta(a,b)[a,b]$$
$$[a,m]' = \theta_2(a)[a,m]$$
$$[m,a]' = \theta_1(a)[m,a]$$
$$[m,m,m]' = \phi(m)[m,m,m]$$
$$[m,m,a]' = \phi(a)[m,m,a]$$

$$\theta(a,b), \theta_1(a), \theta_2(a), \phi(a) \in k^\times$$
$$\phi(m) \in GL(V)$$

respects the assumption

$$\alpha_3(a,b) = 1, \beta_2(a,1) = 1, \beta_1(a,1) = 1$$

if and only if

$$\theta_1(a) = \theta_2(a) = \frac{\phi(1)}{\phi(a)}$$
$$\theta(a,b) = \frac{\phi(a)\phi(b)}{\phi(1)\phi(ab)}.$$

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3. Second reduction. We make here additional assumptions:

(i) $\alpha_2(a,b) = 1$ for all $a, b \in A$.

(ii) The representation of A on V given by $a \mapsto \beta_2(a,m)$ is a sum of distinct one-dimensional representations, not including the trivial representation.

These are satisfied in the case of the one-dimensional affine transformation groups as we saw in Section 5.

(ii) means that V has a basis $\{l_x \mid x \in X\}$ indexed by a subset X of the character group $\hat{A} = \operatorname{Hom}(A, k^\times)$ with $1 \notin X$ so that

$$\beta_2(a,m) \colon l_x \mapsto x(a) l_x$$

for all $a \in A$.

Then $\gamma(1) \colon V \to V$ should be of the form

$$\gamma(1) \colon l_x \mapsto s_x l_{\sigma(x)}$$

where $s_x \in k^\times$ and $\sigma \colon X \to X$ is a bijection.

Using the basis $\{l_x\}$, we write

$$\gamma(m,m) \colon l_x \otimes l_y \mapsto \sum_{u,v} p_{x,y}^{u,v} l_u \otimes l_v$$

$$\gamma(m,1) \colon 1 \mapsto \sum_{u,v} q_{u,v} l_u \otimes l_v$$

$$\gamma(1,m) \colon l_x \otimes l_y \mapsto r_{x,y}$$

with scalars $p_{x,y}^{u,v}, q_{u,v}, r_{x,y}$.

Then the equations of Section 7.1 reduce to the following (i)–(xi):

(i) A is abelian.

(ii) $X = \hat{A} - \{1\}$.

(iii)

$$\sigma^3 = 1$$

$$\sigma(x^{-1})\sigma^{-1}(x) = 1$$

$$
\begin{array}{ccc}
\sigma(x)y = u & & x = \sigma^{-1}(u)v \\
\sigma^{-1}(x)\sigma^{-1}(y) = \sigma^{-1}(v) & \Longleftrightarrow & \sigma(y) = \sigma(u)\sigma(v)
\end{array}
$$

(iv)

$$\beta_1(a,m) \colon l_x \mapsto \sigma^{-1}(x)(a^{-1}) l_x$$

$$\beta_2(a,m) \colon l_x \mapsto x(a) l_x$$

$$\beta_3(a,m) \colon l_x \mapsto \sigma(x)(a^{-1}) l_x$$

(v)

$$\gamma(m,m) \colon l_x \otimes l_y \mapsto \sum_{u,v} p_{x,y}^{u,v} l_u \otimes l_v$$

$$\gamma(1,m) \colon l_x \otimes l_y \mapsto r_{x,y}$$

$$\gamma(m,1) \colon 1 \mapsto \sum_{u,v} q_{u,v} l_u \otimes l_v$$

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with

$$p^{u,v}_{x,y} \neq 0 \iff \begin{array}{c} \sigma(x)y = u \\ \sigma^{-1}(x)\sigma^{-1}(y) = \sigma^{-1}(v) \end{array} \iff \begin{array}{c} x = \sigma^{-1}(u)v \\ \sigma(y) = \sigma(u)\sigma(v) \end{array}$$

$$r_{x,y} \neq 0 \iff \sigma(x)y = 1 \iff x\sigma(y) = 1 \iff \sigma^{-1}(x)\sigma^{-1}(y) = 1$$

$$q_{u,v} \neq 0 \iff \sigma^{-1}(u)v = 1 \iff u\sigma^{-1}(v) = 1 \iff \sigma(u)\sigma(v) = 1$$

(vi)

$$\gamma(1,1) = \frac{\epsilon}{|A|}$$
$$\epsilon = \pm 1$$

(vii)

$$\gamma(1) \colon l_x \mapsto s_x l_{\sigma(x)}$$

(viii)

$$s_x s_{\sigma(x)} s_{\sigma^2(x)} = \epsilon$$

(ix)

$$q_{y,x} = \epsilon q_{x,y}$$
$$r_{y,x} = \epsilon r_{x,y}$$
$$q_{\sigma^{-1}(x),y} r_{x,y} = \frac{1}{|A|} \frac{1}{s_{\sigma^{-1}(x)}}$$
$$\frac{q_{\sigma(x),y}}{q_{x,\sigma(y)}} = \frac{s_x}{s_y}$$
$$\frac{r_{x,\sigma(y)}}{r_{\sigma(x),y}} = \frac{s_x}{s_y}$$

(x) Write $\tau(x) = x^{-1}$ for $x \in X$.

$$p^{u,v}_{x,y} = p^{\sigma\tau(y),u}_{\sigma(x),\sigma\tau(v)} s_x \frac{q_{\sigma\tau(v),v}}{q_{\sigma\tau(y),y}}$$

$$p^{u,v}_{x,y} = p^{\sigma(u),\tau\sigma(x)}_{y,\tau\sigma(v)} \frac{1}{s_u} \frac{r_{x,\tau\sigma(x)}}{r_{v,\tau\sigma(v)}}$$

$$p^{u,v}_{x,y} = \frac{1}{p^{y,\sigma\tau(v)}_{\sigma\tau(x),u}} \frac{q_{v,\sigma\tau(v)}}{q_{x,\sigma\tau(x)}}$$

$$p^{u,v}_{x,y} = \frac{1}{p^{\tau\sigma(u),x}_{v,\tau\sigma(y)}} \frac{r_{y,\tau\sigma(y)}}{r_{u,\tau\sigma(u)}}$$

$$p^{u,v}_{x,y} = \frac{1}{p^{\sigma(y),\sigma(x)}_{u,\sigma(v)}} \frac{s_x s_y}{s_v}$$

$$p^{u,v}_{x,y} = \frac{1}{p^{\sigma\tau(u),y}_{\tau\sigma(x),v}} \frac{\epsilon}{s_{\sigma\tau(u)}} \frac{r_{x,\tau\sigma(x)}}{r_{u,\tau\sigma(u)}}$$

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(xi)
$$p_{x,y}^{x',y'} p_{y',z}^{y'',z'} p_{x',y''}^{x'',y'''} = p_{y,z}^{x'',z_1} p_{x,z_1}^{y''',z'}$$

In (ix), (x) and (xi) it is understood that all p, q, r involved are nonzero.
Put
$$Z = \left\{ (x, y, u, v) \in X^4 \; \middle| \; \begin{array}{l} \sigma(x)y = u, \sigma^{-1}(x)\sigma^{-1}(y) = \sigma^{-1}(v) \\ x = \sigma^{-1}(u)v, \sigma(y) = \sigma(u)\sigma(v) \end{array} \right\}.$$

Then the symmetric group S_4 acts on Z as follows:
$$(x, y; u, v) \mapsto (\sigma(x), \sigma\tau(v); \sigma\tau(y), u)$$
$$(x, y; u, v) \mapsto (y, \tau\sigma(v); \sigma(u), \tau\sigma(x))$$
$$(x, y; u, v) \mapsto (\sigma\tau(x), u; y, \sigma\tau(v))$$
$$(x, y; u, v) \mapsto (v, \tau\sigma(y); \tau\sigma(u), x)$$
$$(x, y; u, v) \mapsto (u, \sigma(v); \sigma(y), \sigma(x))$$
$$(x, y; u, v) \mapsto (\tau\sigma(x), v; \sigma\tau(u), y).$$

The equations in (x) are compatible with this action.
Put
$$W_l = \left\{ (x, y, z) \in X^3 \; \middle| \; \begin{array}{r} \sigma(x)y \neq 1 \\ \sigma(y)z \neq 1 \\ \sigma^{-1}(x)\sigma^{-1}(y)\sigma^{-1}(z) \neq 1 \end{array} \right\}.$$

Then we have a bijection
$$\left\{ (x, y, z, x'', y''', z', x', y', y'', z_1) \in X^{10} \; \middle| \; \begin{array}{l} (x, y, x', y') \in Z \\ (y', z, y'', z') \in Z \\ (x', y'', x'', y''') \in Z \\ (y, z, x'', z_1) \in Z \\ (x, z_1, y''', z') \in Z \end{array} \right\} \to W_l$$
$$(x, y, z, x'', y''', z', x', y', y'', z_1) \mapsto (x, y, z).$$

So we have one equation (xi) for each $(x, y, z) \in W_l$.

4. Change of bases. We examine how the base change given by
$$[a, b]' = \theta(a, b)[a, b]$$
$$[a, m]' = \theta_2(a)[a, m]$$
$$[m, a]' = \theta_1(a)[m, a]$$
$$[m, m, m]' = \phi(m)[m, m, m]$$
$$[m, m, a]' = \phi(a)[m, m, a]$$

with
$$\theta(a, b) = \frac{\phi(a)\phi(b)}{\phi(1)\phi(ab)}$$
$$\theta_1(a) = \theta_2(a) = \frac{\phi(1)}{\phi(a)}$$
$$\phi(a) \in k^\times$$
$$\phi(m) \in GL(V)$$

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modifies the parameters $\sigma, s_x, p_{x,y}^{u,v}, q_{u,v}, r_{x,y}$

Let $(l'_{x'})_{x' \in X'}$ be a diagonalizing basis for the representations $\beta'_2(-, m)$ of A on V. The map $\phi(m)\colon V \to V$ is an isomorphism between the representations $\beta_2(-, m)$ and $\beta'_2(-, m)$. So $X' = X$ and $\phi(m)$ is of the form

$$\phi(m)\colon l_x \mapsto \lambda_x l'_x \quad \text{for } x \in X$$

with $\lambda_x \in k^\times$. It turns out that

$$X' = X$$
$$\sigma' = \sigma$$
$$p'^{u,v}_{x,y} = \frac{\lambda_u \lambda_v}{\lambda_x \lambda_y} p_{x,y}^{u,v}$$
$$q'_{u,v} = \frac{\lambda_u \lambda_v}{\phi(1)} q_{u,v}$$
$$r'_{x,y} = \frac{\phi(1)}{\lambda_x \lambda_y} r_{x,y}$$
$$s'_x = \frac{\lambda_{\sigma(x)}}{\lambda_x} s_x$$
$$\epsilon' = \epsilon.$$

8. Recovery of a finite field

THEOREM. *Let B be a finite abelian group, $X = B - \{1\}$. If a map $\sigma\colon X \to X$ has the properties*

$$\sigma^3 = 1$$
$$\sigma(x^{-1})\sigma^{-1}(x) = 1$$

and

$$\begin{array}{cc} \sigma(x)y = u & x = \sigma^{-1}(u)v \\ \sigma^{-1}(x)\sigma^{-1}(y) = \sigma^{-1}(v) & \overset{\Longleftrightarrow}{} \quad \sigma(y) = \sigma(u)\sigma(v) \end{array}$$

for all $x, y, u, v \in X$, then B is the multiplicative group of a field F and $\sigma(x) = 1 - \frac{1}{x}$ for all $x \in X$.

Therefore we will identify $\hat{A} = F^\times$ with F a finite field, and $\sigma(x) = 1 - \frac{1}{x}$. Then

$$\sigma^{-1}(x) = \frac{1}{1-x}$$
$$\tau(x) = \frac{1}{x}$$
$$\sigma\tau(x) = 1 - x$$
$$\tau\sigma(x) = \frac{x}{x-1}.$$

We have $\sigma\tau(x) = x$ iff $2x = 1$. So $\sigma\tau$ has a fixed point iff $\text{char}(F) \neq 2$. If $\sigma\tau(x) = x$, the equation $q_{x,\sigma\tau(x)} = \epsilon q_{\sigma\tau(x),x}$ yields $\epsilon = 1$. Thus

$$\text{char}(F) \neq 2 \implies \epsilon = 1.$$

After a suitable change of bases we may assume

$$s_x = \epsilon \quad \text{for all } x \in X.$$

The base change given by $(\lambda_x)_x$ respects this assumption iff

$$\lambda_{\sigma(x)} = \lambda_x.$$

Equation (ix) now becomes

$$q_{y,x} = \epsilon q_{x,y}$$
$$r_{y,x} = \epsilon r_{x,y}$$
$$q_{\sigma^{-1}(x),y} r_{x,y} = \frac{\epsilon}{|A|}$$
$$q_{\sigma(x),y} = q_{x,\sigma(y)}$$
$$r_{\sigma(x),y} = r_{x,\sigma(y)}.$$

From now on we assume $\text{char}(F) = 2$. Let X_* be a representative system of $\langle \sigma, \tau \rangle$-orbits in X. Since τ has no fixed point in X, τ leaves no $\langle \sigma \rangle$-orbit invariant. Put

$$X_0 = \langle \sigma \rangle \cdot X_*, \quad X_1 = \tau \cdot X_0.$$

Then

$$X = X_0 \cup X_1 \quad \text{(disjoint)}.$$

Make the base change given by

$$\phi(a) = 1,$$
$$\lambda_x = \begin{cases} r_{x,\tau\sigma(x)} & \text{for } x \in X_0 \\ 1 & \text{for } x \in X_1. \end{cases}$$

Then

$$r'_{x,\tau\sigma(x)} = \begin{cases} 1 & \text{for } x \in X_0 \\ \epsilon & \text{for } x \in X_1 \end{cases}$$
$$q'_{x,\sigma\tau(x)} = \begin{cases} \frac{\epsilon}{|A|} & \text{for } x \in X_0 \\ \frac{1}{|A|} & \text{for } x \in X_1. \end{cases}$$

Thus we may assume

$$r_{x,\tau\sigma(x)} = \begin{cases} 1 & \text{for } x \in X_0 \\ \epsilon & \text{for } x \in X_1 \end{cases}$$
$$q_{x,\sigma\tau(x)} = \begin{cases} \frac{\epsilon}{|A|} & \text{for } x \in X_0 \\ \frac{1}{|A|} & \text{for } x \in X_1. \end{cases}$$

The base change given by (λ_x) respects this assumption iff

$$\lambda_x = \lambda_{\sigma(x)}$$
$$\lambda_x \lambda_{\tau(x)} = \phi(1).$$

9. Small finite fields

1. Case of $\mathbb{F}_8$. Here we let

$$A = (\mathbb{F}_8^\times)\hat{\ }$$
$$X = A\hat{\ } - \{1\} = \mathbb{F}_8 - \{0,1\}$$
$$\sigma(x) = 1 - \tfrac{1}{x}.$$

We have $\mathbb{F}_8 = \mathbb{F}_2(\alpha)$ with $\alpha^3 + \alpha + 1 = 0$, $\alpha^7 = 1$. Then $\mathbb{F}_8^\times = \langle\alpha\rangle$ and $X = \{\alpha^i \mid i = 1, \ldots, 6\}$. The cycle presentation of σ is

$$(\alpha^1\alpha^2\alpha^4)(\alpha^3\alpha^5\alpha^6).$$

$\langle\sigma,\tau\rangle$ acts transitively on X. Put

$$X_0 = \{\alpha^1, \alpha^2, \alpha^4\}, \quad X_1 = \tau(X_0) = \{\alpha^6, \alpha^5, \alpha^3\}.$$

Then

$$X = X_0 \cup X_1.$$

As in the previous section we may assume

$$s_x = \epsilon$$
$$r_{x,\tau\sigma(x)} = \begin{cases} 1 & \text{for } x \in X_0 \\ \epsilon & \text{for } x \in X_1 \end{cases}$$
$$q_{x,\sigma\tau(x)} = \begin{cases} \frac{\epsilon}{|A|} & \text{for } x \in X_0 \\ \frac{1}{|A|} & \text{for } x \in X_1. \end{cases}$$

The base change by $\phi(a), (\lambda_x)_x$ respects this assumption iff

$$\lambda_x = \lambda_{\sigma(x)}$$
$$\lambda_x\lambda_{\tau(x)} = \phi(1).$$

The set Z consists of 30 elements

$$(\alpha^1, \alpha^1; \alpha^3, \alpha^2), \quad (\alpha^1, \alpha^2; \alpha^4, \alpha^6), \quad (\alpha^1, \alpha^3; \alpha^5, \alpha^5),$$
$$(\alpha^1, \alpha^4; \alpha^6, \alpha^3), \quad (\alpha^1, \alpha^6; \alpha^1, \alpha^4), \quad (\alpha^2, \alpha^1; \alpha^5, \alpha^6),$$
$$(\alpha^2, \alpha^2; \alpha^6, \alpha^4), \quad (\alpha^2, \alpha^4; \alpha^1, \alpha^5), \quad (\alpha^2, \alpha^5; \alpha^2, \alpha^1),$$
$$(\alpha^2, \alpha^6; \alpha^3, \alpha^3), \quad (\alpha^3, \alpha^1; \alpha^6, \alpha^5), \quad (\alpha^3, \alpha^3; \alpha^1, \alpha^6),$$
$$(\alpha^3, \alpha^4; \alpha^2, \alpha^2), \quad (\alpha^3, \alpha^5; \alpha^3, \alpha^4), \quad (\alpha^3, \alpha^6; \alpha^4, \alpha^1),$$
$$(\alpha^4, \alpha^1; \alpha^2, \alpha^3), \quad (\alpha^4, \alpha^2; \alpha^3, \alpha^5), \quad (\alpha^4, \alpha^3; \alpha^4, \alpha^2),$$
$$(\alpha^4, \alpha^4; \alpha^5, \alpha^1), \quad (\alpha^4, \alpha^5; \alpha^6, \alpha^6), \quad (\alpha^5, \alpha^2; \alpha^1, \alpha^1),$$
$$(\alpha^5, \alpha^3; \alpha^2, \alpha^4), \quad (\alpha^5, \alpha^4; \alpha^3, \alpha^6), \quad (\alpha^5, \alpha^5; \alpha^4, \alpha^3),$$
$$(\alpha^5, \alpha^6; \alpha^5, \alpha^2), \quad (\alpha^6, \alpha^1; \alpha^4, \alpha^4), \quad (\alpha^6, \alpha^2; \alpha^5, \alpha^3),$$
$$(\alpha^6, \alpha^3; \alpha^6, \alpha^1), \quad (\alpha^6, \alpha^5; \alpha^1, \alpha^2), \quad (\alpha^6, \alpha^6; \alpha^2, \alpha^5).$$

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Name them $z_1, \ldots, z_{30}$ in this order.

S_4 acts on the set Z and Z is divided to two S_4-orbits B_1, B_2.

$$B_1 = \{z_1, z_2, z_3, z_5, z_7, z_8, z_9, z_{10}, z_{11}, z_{12}, z_{13}, z_{14},$$
$$z_{16}, z_{18}, z_{19}, z_{20}, z_{21}, z_{23}, z_{24}, z_{25}, z_{26}, z_{27}, z_{28}, z_{30}\},$$
$$B_2 = \{z_4, z_6, z_{15}, z_{17}, z_{22}, z_{29}\}.$$

A representative system of S_4-orbits in Z is given by $\{z_1, z_4\}$.

We write $p_{x,y}^{u,v} = p(x, y; u, v)$ as well. By (x) of the preceding section, $p(z)$ for $z' \in Z - \{z_1, z_4\}$ are expressed by $p(z_1)$, $p(z_4)$.

We have 120 equations (xi) corresponding to $(x, y, z) \in W_l$.

$$p_{x,y}^{x',y'} p_{y',z}^{y'',z'} p_{x',y''}^{x'',y'''} = p_{y,z}^{x'',z_1} p_{x,z_1}^{y''',z'}.$$

After substitution of the expressions of $p(z)$ by $p(z_1)$, $p(z_4)$, they reduce to a single equation

$$p(z_4) = p(z_1)^2.$$

Recall that $\langle \sigma, \tau \rangle$ acts regularly on X. So, for any nonzero scalar l we have a unique function $\lambda \colon X \to k^\times$ with property

$$\lambda_{\alpha^1} = l$$
$$\lambda_{\sigma(x)} = \lambda_x = \lambda_{\tau(x)}^{-1} \quad \text{for all } x \in X.$$

The base transformation given by $\phi(a) = 1$ and $(\lambda_x)_x$ has the effect

$$p'(z_1) = p(z_1)l^{-2}$$
$$p'(z_4) = p(z_4)l^{-4}$$

So taking $l = p(z_1)^{\frac{1}{2}}$, we may assume

$$p(z_1) = 1.$$

Thus $p(z) = 1$ for

$$z = z_1, z_2, z_4, z_6, z_7, z_8, z_{12}, z_{14}, z_{16}, z_{17}, z_{19}, z_{24}, z_{25}, z_{28}, z_{30},$$

while $p(z) = \epsilon$ for

$$z = z_3, z_5, z_9, z_{10}, z_{11}, z_{13}, z_{15}, z_{18}, z_{20}, z_{21}, z_{22}, z_{23}, z_{26}, z_{27}, z_{29}.$$

Thus after a suitable base transformation, the pentagon equations have the two solutions depending on the values of $\epsilon = \pm 1$:

$$s_x = \epsilon$$
$$r_{\alpha^1, \alpha^5} = r_{\alpha^2, \alpha^3} = r_{\alpha^4, \alpha^6} = 1$$
$$r_{\alpha^5, \alpha^1} = r_{\alpha^3, \alpha^2} = r_{\alpha^6, \alpha^4} = \epsilon$$
$$q_{\alpha^1, \alpha^3} = q_{\alpha^2, \alpha^6} = q_{\alpha^4, \alpha^5} = \frac{\epsilon}{|A|}$$
$$q_{\alpha^3, \alpha^1} = q_{\alpha^6, \alpha^2} = q_{\alpha^5, \alpha^4} = \frac{1}{|A|}.$$

$p(z) = 1$ for

$$z = (\alpha^1, \alpha^1, \alpha^3, \alpha^2), \quad (\alpha^1, \alpha^2, \alpha^4, \alpha^6), \quad (\alpha^1, \alpha^4, \alpha^6, \alpha^3),$$
$$(\alpha^2, \alpha^1, \alpha^5, \alpha^6), \quad (\alpha^2, \alpha^2, \alpha^6, \alpha^4), \quad (\alpha^2, \alpha^4, \alpha^1, \alpha^5),$$
$$(\alpha^3, \alpha^3, \alpha^1, \alpha^6), \quad (\alpha^3, \alpha^5, \alpha^3, \alpha^4), \quad (\alpha^4, \alpha^1, \alpha^2, \alpha^3),$$
$$(\alpha^4, \alpha^2, \alpha^3, \alpha^5), \quad (\alpha^4, \alpha^4, \alpha^5, \alpha^1), \quad (\alpha^5, \alpha^5, \alpha^4, \alpha^3),$$
$$(\alpha^5, \alpha^6, \alpha^5, \alpha^2), \quad (\alpha^6, \alpha^3, \alpha^6, \alpha^1), \quad (\alpha^6, \alpha^6, \alpha^2, \alpha^5).$$

$p(z) = \epsilon$ for

$$z = (\alpha^1, \alpha^3, \alpha^5, \alpha^5), \quad (\alpha^1, \alpha^6, \alpha^1, \alpha^4), \quad (\alpha^2, \alpha^5, \alpha^2, \alpha^1),$$
$$(\alpha^2, \alpha^6, \alpha^3, \alpha^3), \quad (\alpha^3, \alpha^1, \alpha^6, \alpha^5), \quad (\alpha^3, \alpha^4, \alpha^2, \alpha^2),$$
$$(\alpha^3, \alpha^6, \alpha^4, \alpha^1), \quad (\alpha^4, \alpha^3, \alpha^4, \alpha^2), \quad (\alpha^4, \alpha^5, \alpha^6, \alpha^6),$$
$$(\alpha^5, \alpha^2, \alpha^1, \alpha^1), \quad (\alpha^5, \alpha^3, \alpha^2, \alpha^4), \quad (\alpha^5, \alpha^4, \alpha^3, \alpha^6),$$
$$(\alpha^6, \alpha^1, \alpha^4, \alpha^4), \quad (\alpha^6, \alpha^2, \alpha^5, \alpha^3), \quad (\alpha^6, \alpha^5, \alpha^1, \alpha^2).$$

2. Case of $\mathbb{F}_4$. Let

$$A = (\mathbb{F}_4^\times)\hat{\ }$$
$$X = A\hat{\ } - \{1\} = \mathbb{F}_4 - \{0, 1\}$$
$$\sigma(x) = 1 - \tfrac{1}{x}.$$

We have $\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2\}$ with $\alpha^2 + \alpha + 1 = 0$, $\alpha^3 = 1$. Then $X = \{\alpha, \alpha^2\}$, $\sigma = 1$, and
$$Z = \{(\alpha, \alpha; \alpha^2, \alpha^2), (\alpha^2, \alpha^2; \alpha, \alpha)\}.$$

We may assume

$$s_x = \epsilon$$
$$r_{\alpha, \alpha^2} = 1$$
$$r_{\alpha^2, \alpha} = \epsilon$$
$$q_{\alpha, \alpha^2} = \tfrac{\epsilon}{|A|}$$
$$q_{\alpha^2, \alpha} = \tfrac{1}{|A|}.$$

Then equation (x) reduce to

$$p_{\alpha, \alpha}^{\alpha^2, \alpha^2} \, p_{\alpha^2, \alpha^2}^{\alpha, \alpha} = \epsilon.$$

This time
$$W_l = \{(x, y, z) \in X^3 \mid xy \neq 1, yz \neq 1, xyz \neq 1\} = \emptyset,$$

so there is no equation (xi).

Base transformation
$$\phi(1) = 1, \lambda_\alpha = l, \lambda_{\alpha^2} = l^{-1}$$

yields

$$p'^{\alpha^2,\alpha^2}_{\alpha,\alpha} = l^{-4} p^{\alpha^2,\alpha^2}_{\alpha,\alpha}.$$

So we can take l so that

$$p^{\alpha^2,\alpha^2}_{\alpha,\alpha} = 1.$$

Thus the pentagon equation have two solutions depending on $\epsilon = \pm 1$.

$$s_x = \epsilon$$
$$r_{\alpha,\alpha^2} = 1$$
$$r_{\alpha^2,\alpha} = \epsilon$$
$$q_{\alpha,\alpha^2} = \frac{\epsilon}{|A|}$$
$$q_{\alpha^2,\alpha} = \frac{1}{|A|}$$
$$p^{\alpha^2,\alpha^2}_{\alpha,\alpha} = 1$$
$$p^{\alpha,\alpha}_{\alpha^2,\alpha^2} = \epsilon.$$

It can be checked that when the assumptions made in Sections 7.1 and 7.3 are not satisfied, there is no solution. Thus the above solution with $\epsilon = -1$ is the unique nontrivial deformation of $\mathrm{Rep}(\mathbb{F}_4 \rtimes \mathbb{F}_4^\times)$.

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