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研究組織

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研究発表

学会誌等

1. D. Tambara, A duality for modules over monoidal categories of representations of semisimple Hopf algebras, J. Alg. 241 (2001).
2. D. Tambara, Invariants and semi-direct products for finite group actions on tensor categorires, J. Matn. Soc. Japan 53 (2001).

研究成果による工業所有権の出願取得状況

なし

0. introduction

For a tensor category \mathcal{A} over a field k , the category ${}_{\mathcal{A}}\mathbb{B}(\mathcal{A}, \mathcal{A})_{\mathcal{A}}$ of functors $\mathcal{A}^{\text{op}} \otimes \mathcal{A} \rightarrow \{\text{vector spaces}\}$ with two-sided tensor actions is defined. When \mathcal{A} is semi-simple, ${}_{\mathcal{A}}\mathbb{B}(\mathcal{A}, \mathcal{A})_{\mathcal{A}}$ is equivalent to the center of \mathcal{A} . As an interesting example of non semi-simple tensor categories, we take \mathcal{A} to be the Mackey category \mathcal{M} of a finite group G . The category of G -sets is a subcategory of \mathcal{M} and the category of permutation G -modules is a quotient of \mathcal{M} . Our result is that the category ${}_{\mathcal{M}}\mathbb{B}(\mathcal{M}, \mathcal{M})_{\mathcal{M}}$ is equivalent to the category of Mackey functors on the category of connected G -sets equipped with automorphisms. (Theorem 12.6).

The following notations are used. \mathcal{V} is the category of vector spaces over k . \mathcal{S} is the category of finite G -sets. For a k -linear categories \mathcal{X} and \mathcal{Y} , $\text{Hom}(\mathcal{X}, \mathcal{Y})$ denotes the category of k -linear functors $\mathcal{X} \rightarrow \mathcal{Y}$.

1. categories $\mathbb{B}(\mathcal{X}, \mathcal{Y})$, ${}_{\mathcal{A}}\mathbb{B}(\mathcal{X}, \mathcal{Y})$, ${}_{\mathcal{A}}\mathbb{B}(\mathcal{X}, \mathcal{Y})_{\mathcal{A}}$

For k -categories \mathcal{X} and \mathcal{Y} , we denote by $\mathbb{B}(\mathcal{X}, \mathcal{Y})$ the category of k -bilinear functors $\mathcal{X}^{\text{op}} \times \mathcal{Y} \rightarrow \mathcal{V}$. Namely an object of $\mathbb{B}(\mathcal{X}, \mathcal{Y})$ consists of k -spaces $\phi(X, Y)$ for all objects X in \mathcal{X} and Y in \mathcal{Y} , and k -linear maps $[f, g]: \phi(X', Y) \rightarrow \phi(X, Y')$ for all morphisms $f: X \rightarrow X'$ in \mathcal{X} and $g: Y \rightarrow Y'$ in \mathcal{Y} satisfying the following conditions.

(1.1.i) For morphisms $f: X' \rightarrow X$, $f': X'' \rightarrow X'$ in \mathcal{X} and $g: Y \rightarrow Y'$ and $g': Y' \rightarrow Y''$, we have

$$[f \circ f', g' \circ g] = [f', g'] \circ [f, g]$$

(1.1.ii)

$$[1, 1] = 1$$

(1.1.iii) $[f, g]$ is bilinear in f and g .

Easy consequences of these conditions are

$$[f, g] = [f, 1] \circ [1, g] = [1, g] \circ [f, 1]$$

and

$$\phi(X_1 \oplus X_2, Y) \cong \phi(X_1, Y) \oplus \phi(X_2, Y)$$

$$\phi(X, Y_1 \oplus Y_2) \cong \phi(X, Y_1) \oplus \phi(X, Y_2)$$

Let \mathcal{A} be a tensor category. We assume that \mathcal{A} is strict. The tensor product of X and Y is denoted by XY . The tensor product of morphisms $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ is denoted by $fg: XY \rightarrow X'Y'$, while the composition of $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ is denoted by $g \circ f: X \rightarrow Z$. The unit object of \mathcal{A} is denoted by I .

We have a notion of \mathcal{A} -modules. See [2] for the definition. Let \mathcal{X}, \mathcal{Y} left \mathcal{A} -modules. We define the category ${}_{\mathcal{A}}\mathbb{B}(\mathcal{X}, \mathcal{Y})$ as follows. An object of ${}_{\mathcal{A}}\mathbb{B}(\mathcal{X}, \mathcal{Y})$ is an object ϕ in $\mathbb{B}(\mathcal{X}, \mathcal{Y})$ equipped with linear maps

$$A.: \phi(X, Y) \rightarrow \phi(AX, AY)$$

for all objects A in \mathcal{A} , X in \mathcal{X} , and Y in \mathcal{Y} , satisfying the following conditions.

(1.2.i) For morphisms $f: X' \rightarrow X$ in \mathcal{X} and $g: Y \rightarrow Y'$ in \mathcal{Y} , we have a commutative diagram

$$\begin{array}{ccc} \phi(X, Y) & \xrightarrow{A.} & \phi(AX, AY) \\ [f, g] \downarrow & & \downarrow [Af, Ag] \\ \phi(X', Y') & \xrightarrow{A.} & \phi(AX', AY') \end{array}$$

(1.2.ii) For a morphism $h: A \rightarrow A'$ in \mathcal{A} , we have a commutative diagram

$$\begin{array}{ccc} \phi(X, Y) & \xrightarrow{A.} & \phi(AX, AY) \\ A'. \downarrow & & \downarrow [1, hY] \\ \phi(A'X, A'Y) & \xrightarrow{[hX, 1]} & \phi(AX, A'Y) \end{array}$$

(1.2.iii) For objects A, A' in \mathcal{A} , we have a commutative diagram

$$\begin{array}{ccc} \phi(X, Y) & \xrightarrow{A.} & \phi(AX, AY) \\ (A'A). \searrow & & \downarrow A'. \\ & & \phi(A'AX, A'AY) \end{array}$$

(1.2.iv) For the unit object I , $I.: \phi(X, Y) \rightarrow \phi(X, Y)$ is the identity.

Let \mathcal{X}, \mathcal{Y} be $(\mathcal{A}, \mathcal{A})$ -bimodules. The category ${}_{\mathcal{A}}\mathbb{B}(\mathcal{X}, \mathcal{Y})_{\mathcal{A}}$ is defined as follows. An object of ${}_{\mathcal{A}}\mathbb{B}(\mathcal{X}, \mathcal{Y})_{\mathcal{A}}$ is an object ϕ in $\mathbb{B}(\mathcal{X}, \mathcal{Y})$ equipped with linear maps

$$\begin{aligned} A.: \phi(X, Y) &\rightarrow \phi(AX, AY) \\ .A: \phi(X, Y) &\rightarrow \phi(XA, YA) \end{aligned}$$

satisfying the following conditions.

- (1.3.i) same as (1.2.i).
- (1.3.ii) same as (1.2.ii).
- (1.3.iii) same as (1.2.iii).
- (1.3.iv) same as (1.2.iv).
- (1.3.v) The analogue of (1.2.i) for $.A$.
- (1.3.vi) The analogue of (1.2.ii) for $.A$.
- (1.3.vii) The analogue of (1.2.iii) for $.A$.
- (1.3.viii) The analogue of (1.2.iv) for $.I$.
- (1.3.ix) For objects A, B in \mathcal{A} and X in \mathcal{X} and Y in \mathcal{Y} , the diagram

$$\begin{array}{ccc} \phi(X, Y) & \xrightarrow{A.} & \phi(AX, AY) \\ .B \downarrow & & \downarrow .B \\ \phi(XB, YB) & \xrightarrow{A.} & \phi(AXB, AXB) \end{array}$$

is commutative.

2. category ${}_{\mathcal{A}}\mathbb{B}(\mathcal{A}, \mathcal{X})$

We call a quadruple (A, A', ϵ, η) a duality if A, A' are objects in \mathcal{A} and $\epsilon: AA' \rightarrow I, \eta: I \rightarrow A'A$ are morphisms in \mathcal{A} such that the following diagrams commute.

$$\begin{array}{ccc} A & \xrightarrow{A\eta} & AA'A \\ 1 \searrow & & \downarrow \epsilon A \\ & & A \end{array}, \quad \begin{array}{ccc} A' & \xrightarrow{\eta A'} & A'AA' \\ 1 \searrow & & \downarrow A'\epsilon \\ & & A' \end{array}$$

It is well-known that a duality (A, A', ϵ, η) gives rise to the adjoint isomorphisms

$$\text{Hom}(AB, C) \cong \text{Hom}(A, A'C)$$

for B, C in \mathcal{A} , and vice versa.

We will show that such an isomorphism holds also for any object ϕ in ${}_{\mathcal{A}}\mathbb{B}(\mathcal{X}, \mathcal{Y})$.

Proposition 2.1. *Let ϕ be an object in ${}_{\mathcal{A}}\mathbb{B}(\mathcal{X}, \mathcal{Y})$, and (A, A', ϵ, η) a duality in \mathcal{A} . Then we have natural isomorphisms*

$$\phi(AX, Y) \cong \phi(X, A'Y)$$

for all $X \in \mathcal{X}, Y \in \mathcal{Y}$.

Proof. Define the maps σ, τ to be the composites

$$\begin{aligned} \sigma: \phi(AX, Y) &\xrightarrow{A'} \phi(A'AX, A'Y) \xrightarrow{[\eta X, 1]} \phi(X, A'Y) \\ \tau: \phi(X, A'Y) &\xrightarrow{A} \phi(AX, AA'Y) \xrightarrow{[1, \epsilon Y]} \phi(AX, Y) \end{aligned}$$

We will show they are inverse to each other.

We have a commutative diagram

$$\begin{array}{ccccc} \phi(AX, Y) & \xrightarrow{A'} & \phi(A'AX, A'Y) & \xrightarrow{A} & \phi(AA'AX, AA'Y) \\ \sigma \searrow & & [\eta X, 1] \downarrow & & \downarrow [A\eta X, 1] \\ & & \phi(X, A'Y) & \xrightarrow{A} & \phi(AX, AA'Y) \\ & & & \tau \searrow & \downarrow [1, \epsilon Y] \\ & & & & \phi(AX, Y) \end{array}$$

Hence

$$\begin{array}{ccc} \phi(AX, Y) & \xrightarrow{(AA')} & \phi(AA'AX, AA'Y) \\ \tau \circ \sigma \searrow & & \downarrow [A\eta X, \epsilon Y] \\ & & \phi(AX, Y) \end{array}$$

But we have

$$\begin{array}{ccc} \phi(AX, Y) & \xrightarrow{(AA')} & \phi(AA'AX, AA'Y) \\ 1 \downarrow & [\epsilon AX, 1] \searrow & \downarrow [1, \epsilon Y] \\ \phi(AX, Y) & \xleftarrow{[A\eta X, 1]} & \phi(AA'AX, Y) \end{array}$$

Hence $\tau \circ \sigma = 1$.

We have a commutative diagram

$$\begin{array}{ccccc}
\phi(X, A'Y) & \xrightarrow{A'} & \phi(AX, AA'Y) & \xrightarrow{A'} & \phi(A'AX, A'AA'Y) \\
\tau \searrow & & \downarrow [1, \eta Y] & & \downarrow [1, A' \eta Y] \\
& & \phi(AX, Y) & \xrightarrow{A'} & \phi(A'AX, A'Y) \\
& & \sigma \searrow & & \downarrow [\epsilon X, 1] \\
& & & & \phi(X, A'Y)
\end{array}$$

Hence

$$\begin{array}{ccc}
\phi(X, A'Y) & \xrightarrow{(A'A).} & \phi(A'AX, A'AA'Y) \\
\sigma \circ \tau \searrow & & \downarrow [\eta X, A' \epsilon Y] \\
& & \phi(X, A'Y)
\end{array}$$

But

$$\begin{array}{ccc}
\phi(X, A'Y) & \xrightarrow{(A'A).} & \phi(A'AX, A'AA'Y) \\
1 \downarrow & [1, \eta A'Y] \searrow & \downarrow [\epsilon X, 1] \\
\phi(X, A'Y) & \xleftarrow{[1, A' \epsilon Y]} & \phi(X, A'AA'Y)
\end{array}$$

Hence $\sigma \circ \tau = 1$. This proves the proposition.

Proposition 2.2. *The following diagram is commutative.*

$$\begin{array}{ccc}
\phi(X, Y) & & \\
A. \downarrow & \searrow [1, \epsilon Y] & \\
\phi(AX, AY) & \xrightarrow{\sigma} & \phi(X, A'AY)
\end{array}$$

Proof. This follows from the diagram

$$\begin{array}{ccccc}
& \phi(X, Y) & & & \\
& A. \swarrow & \downarrow (A'A). & \searrow [1, \epsilon Y] & \\
\phi(AX, AY) & \xrightarrow{A'} & \phi(A'AX, A'AY) & \xrightarrow{[\epsilon X, 1]} & \phi(X, A'AY)
\end{array}$$

We say \mathcal{A} is rigid if for every object A in \mathcal{A} there is a duality (A, A', ϵ, η) . Hereafter \mathcal{A} is assumed to be rigid and we choose a duality $(A, A^*, \epsilon_A, \eta_A)$ for each A . Especially for the unit object I we choose the duality $(I, I, 1, 1)$.

For a morphism $f: A \rightarrow B$ in \mathcal{A} , define the morphism $f^*: B^* \rightarrow A^*$ by the commutative diagram

$$\begin{array}{ccc}
B^* & \xrightarrow{f^*} & A^* \\
\eta_A B^* \downarrow & & \uparrow A^* \epsilon_B \\
A^* A B^* & \xrightarrow{A^* f B^*} & A^* B B^*
\end{array}$$

Then the assignments $A \mapsto A^*$ and $f \mapsto f^*$ form a functor $\mathcal{A}^{\text{op}} \rightarrow \mathcal{A}$.

For a morphism $f: A \rightarrow B$ in \mathcal{A} , the following diagrams are commutative.

$$\begin{array}{ccc} I & \xrightarrow{\eta_A} & A^*A \\ \eta_B \downarrow & & \downarrow A^*f \\ B^*B & \xrightarrow{f^*B} & A^*B \end{array} \quad \begin{array}{ccc} AB^* & \xrightarrow{Af^*} & AA^* \\ fB^* \downarrow & & \downarrow \epsilon_A \\ BB^* & \xrightarrow{\epsilon_B} & I \end{array}$$

For objects A and B in \mathcal{A} , we have an isomorphism $\chi_{A,B}: (AB)^* \rightarrow B^*A^*$ such that the following diagrams are commutative.

$$\begin{array}{ccc} I & \xrightarrow{\eta_B} & B^*B \\ \eta_{AB} \downarrow & & \downarrow B^*\eta_{AB} \\ (AB)^*AB & \xrightarrow{\chi_{A,B}AB} & B^*A^*AB \end{array} \quad \begin{array}{ccc} ABB^*A^* & \xleftarrow{AB\chi_{A,B}} & AB(AB)^* \\ A\epsilon_B A^* \downarrow & & \downarrow \epsilon_{AB} \\ AA^* & \xrightarrow{\epsilon_A} & I \end{array}$$

Now we consider ${}_{\mathcal{A}}\mathbb{B}(\mathcal{X}, \mathcal{Y})$ for $\mathcal{X} = \mathcal{A}$ regarded as the left \mathcal{A} -module by left tensor product.

Theorem 2.3. *For any left \mathcal{A} -module \mathcal{Y} , we have an equivalence*

$${}_{\mathcal{A}}\mathbb{B}(\mathcal{A}, \mathcal{Y}) \simeq \text{Hom}(\mathcal{Y}, \mathcal{V}).$$

Proof. We give only the construction of functors. Define the functor

$$S: {}_{\mathcal{A}}\mathbb{B}(\mathcal{A}, \mathcal{Y}) \rightarrow \text{Hom}(\mathcal{Y}, \mathcal{V})$$

by

$$S(\phi)(Y) = \phi(I, Y).$$

Define the functor

$$T: \text{Hom}(\mathcal{Y}, \mathcal{V}) \rightarrow {}_{\mathcal{A}}\mathbb{B}(\mathcal{A}, \mathcal{Y})$$

as follows. For $\psi \in \text{Hom}(\mathcal{Y}, \mathcal{V})$, $\phi = T(\psi)$ is given by

$$\phi(A, Y) = \psi(A^*Y)$$

for objects A in \mathcal{A} and Y in \mathcal{Y} . For morphisms $f: A' \rightarrow A$ and $g: Y \rightarrow Y'$, define

$$[f, g] = \psi(f^*g).$$

For objects A, B in \mathcal{A} and Y in \mathcal{Y} , define

$$B.: \phi(A, Y) \rightarrow \phi(BA, BY)$$

to be the composite

$$\psi(A^*Y) \xrightarrow{\psi(A^*\eta_{BY})} \psi(A^*B^*BY) \xrightarrow{\psi(\chi_{B,A}^{-1}BY)} \psi((BA)^*BY).$$

Then one can verify that S and T are inverse to each other.

3. Mackey category

Let G be a finite group. Let \mathcal{S} be the category of finite G -sets. The direct product of G -sets X and Y is denoted by XY . The direct product of G -maps $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ is denoted by $fg: XY \rightarrow X'Y'$. 1 denotes a one-point G -set.

Let us review the definition of Mackey functors ([1], [3]). A Mackey functor ψ for G consists of vector spaces $\psi(X)$ for all G -sets X and linear maps $f_*: \psi(X) \rightarrow \psi(Y)$ and $f^*: \psi(Y) \rightarrow \psi(X)$ for all G -maps $f: X \rightarrow Y$. They should satisfy the following conditions.

- (3.1.i) $\psi(X)$ and f_* form a functor $\mathcal{S} \rightarrow \mathcal{V}$.
- (3.1.ii) $\psi(X)$ and f^* form a functor $\mathcal{S}^{\text{op}} \rightarrow \mathcal{V}$.
- (3.1.iii) For a pull-back diagram

$$\begin{array}{ccc} X & \xrightarrow{p} & X' \\ f \downarrow & & \downarrow f' \\ Y & \xrightarrow{q} & Y' \end{array}$$

the diagram

$$\begin{array}{ccc} \psi(X) & \xleftarrow{p^*} & \psi(X') \\ f_* \downarrow & & \downarrow f'_* \\ \psi(Y) & \xleftarrow{q^*} & \psi(Y') \end{array}$$

is commutative.

(3.1.iv) Let $i_1: U_1 \rightarrow U_1 + U_2$ and $i_2: U_2 \rightarrow U_1 + U_2$ the inclusion maps. Then the maps

$$\begin{aligned} (i_{1*}, i_{2*}): \psi(U_1) \oplus \psi(U_2) &\rightarrow \psi(U_1 + U_2) \\ (i_1^*, i_2^*): \psi(U_1 + U_2) &\rightarrow \psi(U_1) \oplus \psi(U_2) \end{aligned}$$

are inverse to each other.

(3.1.v) $\psi(\emptyset) = 0$

We denote by $\mathbf{M}(\mathcal{S})$ the category of Mackey functors.

Let us review the definition of the Mackey category for G ([3]). The Mackey category \mathcal{M} is defined as follows. An object of \mathcal{M} is a finite G -set. For G -sets X and Y , the hom-space $\text{Hom}_{\mathcal{M}}(X, Y)$ is the vector space defined by generators and relations as follows. The generators are symbols called "span"

$$[X \leftarrow U \rightarrow Y]$$

for G -maps $U \rightarrow X$ and $U \rightarrow Y$. The relations are

$$[X \leftarrow U_1 + U_2 \rightarrow Y] = [X \leftarrow U_1 \rightarrow Y] + [X \leftarrow U_2 \rightarrow Y].$$

The composition of spans is given by

$$[Y \leftarrow V \rightarrow Z] \circ [X \leftarrow U \rightarrow Y] = [X \leftarrow W \rightarrow Z],$$

where

$$\begin{array}{ccc} W & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & Y \end{array}$$

is a pull-back diagram.

A basic fact is

Theorem 3.2. *We have an equivalence*

$$\mathbb{M}(S) \simeq \text{Hom}(\mathcal{M}, \mathcal{V}).$$

The equivalence is given as follows. For a Mackey functor ψ , define $\psi^\sim: \mathcal{M} \rightarrow \mathcal{V}$ by

$$\psi^\sim(X) = \psi(X)$$

for G -sets X and

$$\psi^\sim([X \xleftarrow{a} U \xrightarrow{b} Y]) = b_* \circ a^*: \psi(X) \rightarrow \psi(Y).$$

Then the assignment $\psi \mapsto \psi^\sim$ yields the equivalence.

The Mackey category \mathcal{M} is a tensor category: The tensor product of G -sets X and Y is just the direct product $X \times Y = XY$. The tensor product of spans are given by

$$[X \leftarrow U \rightarrow Y][X' \leftarrow U' \rightarrow Y'] = [XX' \leftarrow UU' \rightarrow YY'].$$

The unit object is the one-point set $\mathbf{1}$.

The Mackey category is rigid. Indeed, for a G -set X ,

$$(X, X, [XX \leftarrow X \rightarrow \mathbf{1}], [\mathbf{1} \leftarrow X \rightarrow XX])$$

is a duality, where $X \rightarrow XX$ is the diagonal map. This choice of duality yields the functor $(-)^*: \mathcal{M} \rightarrow \mathcal{M}$ given by

$$X^* = X, \quad [X \leftarrow U \rightarrow Y]^* = [Y \leftarrow U \rightarrow X]$$

The isomorphism $\chi_{X,Y}: (XY)^* \rightarrow Y^*X^*$ is just the transposition $XY \rightarrow YX$.

4. categories $\mathbb{M}(S, S)$, ${}_S\mathbb{M}(S, S)$, ${}_S\mathbb{M}(S, S)_S$

In view of the equivalence $\text{Hom}(\mathcal{M}, \mathcal{V}) \simeq \mathbb{M}(S)$ of Theorem 3.2, the categories $\mathbb{B}(\mathcal{M}, \mathcal{M})$, ${}_M\mathbb{B}(\mathcal{M}, \mathcal{M})$, ${}_M\mathbb{B}(\mathcal{M}, \mathcal{M})_M$ are respectively equivalent to the categories $\mathbb{M}(S, S)$, ${}_S\mathbb{M}(S, S)$, ${}_S\mathbb{M}(S, S)_S$ defined below.

The category $\mathbb{M}(S, S)$ is defined as follows. An object ϕ in $\mathbb{M}(S, S)$ consists of k -modules $\phi(X, Y)$ for all G -sets X and Y , and k -linear maps

$$\begin{aligned} \langle f, g \rangle_*: \phi(X, Y) &\rightarrow \phi(X', Y') \\ \langle f, g \rangle^*: \phi(X', Y') &\rightarrow \phi(X, Y) \end{aligned}$$

for all G -maps $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$, which should satisfy the following conditions.

(4.1.i) The collection of $\phi(X, Y)$ and $\langle f, g \rangle_*$ for G -sets X, Y and G -maps f, g forms a functor $\mathcal{S} \times \mathcal{S} \rightarrow \mathcal{V}$.

(4.1.ii) The collection of $\phi(X, Y)$ and $\langle f, g \rangle^*$ for G -sets X, Y and G -maps f, g forms a functor $\mathcal{S}^{\text{op}} \times \mathcal{S}^{\text{op}} \rightarrow \mathcal{V}$.

(4.1.iii) For G -maps $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$, the diagrams

$$\begin{array}{ccc} \phi(X, Y) & \xrightarrow{\langle f, 1 \rangle_*} & \phi(X', Y) & \phi(X, Y) & \xleftarrow{\langle f, 1 \rangle^*} & \phi(X', Y) \\ \langle 1, g \rangle^* \uparrow & & \uparrow \langle 1, g \rangle_* & \langle 1, g \rangle_* \downarrow & & \downarrow \langle 1, g \rangle_* \\ \phi(X, Y') & \xrightarrow{\langle f, 1 \rangle_*} & \phi(X', Y') & \phi(X, Y') & \xleftarrow{\langle f, 1 \rangle^*} & \phi(X', Y') \end{array}$$

are commutative.

(4.1.iv) If

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & X'_1 \\ p \downarrow & & \downarrow p' \\ X_2 & \xrightarrow{f_2} & X'_2 \end{array}$$

is a pull-back diagram, then

$$\begin{array}{ccc} \phi(X_1, Y) & \xrightarrow{\langle f_1, 1 \rangle_*} & \phi(X'_1, Y) \\ \langle p, 1 \rangle^* \uparrow & & \uparrow \langle p', 1 \rangle^* \\ \phi(X_2, Y) & \xrightarrow{\langle f_2, 1 \rangle_*} & \phi(X'_2, Y) \end{array}$$

is commutative.

(4.1.v) If

$$\begin{array}{ccc} Y_1 & \xrightarrow{g_1} & Y'_1 \\ q \downarrow & & \downarrow q' \\ Y_2 & \xrightarrow{g_2} & Y'_2 \end{array}$$

is a pull-back diagram, then

$$\begin{array}{ccc} \phi(X, Y_1) & \xrightarrow{\langle 1, g_1 \rangle_*} & \phi(X, Y'_1) \\ \langle 1, q \rangle^* \uparrow & & \uparrow \langle 1, q' \rangle^* \\ \phi(X, Y_2) & \xrightarrow{\langle 1, g_2 \rangle_*} & \phi(X, Y'_2) \end{array}$$

is commutative.

(4.1.vi) Let $i_1: X_1 \rightarrow X_1 + X_2$, $i_2: X_2 \rightarrow X_1 + X_2$ denote the inclusion maps. Then

$$\begin{aligned} (\langle i_1, 1 \rangle^*, \langle i_2, 1 \rangle^*) &: \phi(X_1 + X_2, Y) \rightarrow \phi(X_1, Y) \oplus \phi(X_2, Y) \\ (\langle i_1, 1 \rangle_*, \langle i_2, 1 \rangle_*) &: \phi(X_1, Y) \oplus \phi(X_2, Y) \rightarrow \phi(X_1 + X_2, Y) \end{aligned}$$

are inverse to each other.

(4.1.vii) Let $i_1: Y_1 \rightarrow Y_1 + Y_2$, $i_2: Y_2 \rightarrow Y_1 + Y_2$ denote the inclusion maps. Then

$$(\langle 1, i_1 \rangle^*, \langle 1, i_2 \rangle^*): \phi(X, Y_1 + Y_2) \rightarrow \phi(X, Y_1) \oplus \phi(X, Y_2)$$

$$(\langle 1, i_1 \rangle_*, \langle 1, i_2 \rangle_*): \phi(X, Y_1) \oplus \phi(X, Y_2) \rightarrow \phi(X, Y_1 + Y_2)$$

are inverse to each other.

$$(4.1.viii) \phi(\emptyset, Y) = 0$$

$$(4.1.ix) \phi(X, \emptyset) = 0$$

A morphism $\sigma: \phi \rightarrow \phi'$ in $\mathbb{M}(\mathcal{S}, \mathcal{S})$ consists of k -linear maps $\sigma_{X,Y}: \phi(X, Y) \rightarrow \phi'(X, Y)$ for all G -sets X and Y , which commute with $\langle f, g \rangle_*$ and $\langle f, g \rangle^*$ for all G -maps f and g .

This ends the definition of $\mathbb{M}(\mathcal{S}, \mathcal{S})$.

We have the equivalence $\mathbb{B}(\mathcal{M}, \mathcal{M}) \simeq \mathbb{M}(\mathcal{S}, \mathcal{S})$.

The category ${}_S\mathbb{M}(\mathcal{S}, \mathcal{S})$ is defined as follows. An object in ${}_S\mathbb{M}(\mathcal{S}, \mathcal{S})$ consists of an object ϕ in $\mathbb{M}(\mathcal{S}, \mathcal{S})$ together with maps

$$Z.: \phi(X, Y) \rightarrow \phi(ZX, ZY)$$

for all G -sets Z , which satisfy the following conditions.

(4.2.i) For any G -maps $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$, the diagrams

$$\begin{array}{ccc} \phi(X, Y) & \xrightarrow{\langle f, g \rangle_*} & \phi(X', Y') \\ Z. \downarrow & & \downarrow Z. \\ \phi(ZX, ZY) & \xrightarrow{\langle Zf, Zg \rangle_*} & \phi(ZX', ZY') \end{array}$$

and

$$\begin{array}{ccc} \phi(X, Y) & \xleftarrow{\langle f, g \rangle^*} & \phi(X', Y') \\ Z. \downarrow & & \downarrow Z. \\ \phi(ZX, ZY) & \xleftarrow{\langle Zf, Zg \rangle^*} & \phi(ZX', ZY') \end{array}$$

are commutative.

(4.2.ii) For any G -map $h: Z \rightarrow Z'$, the diagrams

$$\begin{array}{ccc} \phi(X, Y) & \xrightarrow{Z.} & \phi(ZX, ZY) \\ Z'. \downarrow & & \downarrow \langle 1, hY \rangle_* \\ \phi(Z'X, Z'Y) & \xrightarrow{\langle hX, 1 \rangle_*} & \phi(ZX, Z'Y) \end{array}$$

and

$$\begin{array}{ccc} \phi(X, Y) & \xrightarrow{Z.} & \phi(ZX, ZY) \\ Z'. \downarrow & & \downarrow \langle hX, 1 \rangle_* \\ \phi(Z'X, Z'Y) & \xrightarrow{\langle 1, hY \rangle^*} & \phi(Z'X, ZY) \end{array}$$

commute.

(4.2.iii)

$$\begin{array}{ccc} \phi(X, Y) & \xrightarrow{Z.} & \phi(ZX, ZY) \\ (Z'Z). \searrow & & \downarrow Z'. \\ & & \phi(Z'ZX, Z'ZY) \end{array}$$

(4.2.iv)

$$1.: \phi(X, Y) \rightarrow \phi(1X, 1Y)$$

coincides with the isomorphism

$$\langle p_2, p_2 \rangle^*: \phi(X, Y) \rightarrow \phi(1X, 1Y)$$

where $p_2: 1X \rightarrow X$, $p_2: 1Y \rightarrow Y$ are the second projections.

A morphism $\phi \rightarrow \phi'$ in ${}_S\mathbb{M}(S, S)$ is a morphism $\phi \rightarrow \phi'$ in $\mathbb{M}(S, S)$ making the diagrams

$$\begin{array}{ccc} \phi(X, Y) & \xrightarrow{\sigma_{X,Y}} & \phi'(X, Y) \\ Z. \downarrow & & \downarrow Z. \\ \phi(ZX, ZY) & \xrightarrow{\sigma_{ZX,ZY}} & \phi'(ZX, ZY) \end{array}$$

commutative for all X, Y, Z .

This ends the definition of ${}_S\mathbb{M}(S, S)$.

We have the equivalence ${}_M\mathbb{B}(\mathcal{M}, \mathcal{M}) \simeq {}_S\mathbb{M}(S, S)$.

The category ${}_S\mathbb{M}(S, S)_S$ is defined as follows. An object of this category is an object ϕ of $\mathbb{M}(S, S)$ together with the maps

$$\begin{aligned} Z.: \phi(X, Y) &\rightarrow \phi(ZX, ZY) \\ .Z: \phi(X, Y) &\rightarrow \phi(XZ, YZ) \end{aligned}$$

for all G -sets X, Y, Z satisfying the following conditions.

(4.3.i), (4.3.ii), (4.3.iii), (4.3.iv) : the same as (4.2.i), (4.2.ii), (4.2.iii), (4.2.iv).

(4.3.v), (4.3.vi), (4.3.vii), (4.3.viii): conditions for $.Z$ analogous to (4.2.i), (4.2.ii), (4.2.iii), (4.2.iv).

(4.3.ix) The diagram

$$\begin{array}{ccc} \phi(X, Y) & \xrightarrow{Z.} & \phi(ZX, ZY) \\ .w \downarrow & & \downarrow .w \\ \phi(XW, YW) & \xrightarrow{Z.} & \phi(ZXW, ZYW) \end{array}$$

is commutative.

A morphism $\phi \rightarrow \phi'$ in ${}_S\mathbb{M}(S, S)_S$ is a morphism $\phi \rightarrow \phi'$ in $\mathbb{M}(S, S)$ which commutes with $Z.$ and $.Z$ for all Z .

This ends the definition of ${}_S\mathbb{M}(S, S)_S$.

We have the equivalence ${}_M\mathbb{B}(\mathcal{M}, \mathcal{M})_M \simeq {}_S\mathbb{M}(S, S)_S$.

5. equivalence ${}_S\mathbb{M}(\mathcal{S}, \mathcal{S}) \simeq \mathbb{M}(\mathcal{S})$

By Theorem 2.3 we have an equivalence

$${}_M\mathbb{B}(\mathcal{M}, \mathcal{M}) \simeq \text{Hom}(\mathcal{M}, \mathcal{V}).$$

Since ${}_M\mathbb{B}(\mathcal{M}, \mathcal{M}) \simeq {}_S\mathbb{M}(\mathcal{S}, \mathcal{S})$ and $\text{Hom}(\mathcal{M}, \mathcal{V}) \cong \mathbb{M}(\mathcal{S})$, we obtain

$${}_S\mathbb{M}(\mathcal{S}, \mathcal{S}) \simeq \mathbb{M}(\mathcal{S}).$$

The equivalence is given by the functor

$$S: {}_S\mathbb{M}(\mathcal{S}, \mathcal{S}) \rightarrow \mathbb{M}(\mathcal{S})$$

defined as follows. Let $\phi \in {}_S\mathbb{M}(\mathcal{S}, \mathcal{S})$ and $S(\phi) = \psi$. Then

$$\psi(X) = \phi(1, X)$$

for a G -set X and

$$f_* = \langle 1, f \rangle_*, \quad f^* = \langle 1, f \rangle^*$$

for a G -map f .

The inverse

$$T: \mathbb{M}(\mathcal{S}) \rightarrow {}_S\mathbb{M}(\mathcal{S}, \mathcal{S})$$

of S is given as follows. Let $\psi \in {}_S\mathbb{M}(\mathcal{S}, \mathcal{S})$ and $T(\psi) = \phi$. Then

$$\phi(X, Y) = \psi(XY)$$

for G -sets X, Y , and

$$\langle f, g \rangle_* = (fg)_*, \quad \langle f, g \rangle^* = (fg)^*$$

for G -maps f, g . The operation

$$Z.: \phi(X, Y) \rightarrow \phi(ZX, ZY)$$

is the composite

$$\psi(XY) \xrightarrow{p_{23}^*} \psi(ZXY) \xrightarrow{(p_{12}, p_{13})^*} \psi(ZXZY)$$

where p_{ij} are the projections.

Proposition 2.1 for $\phi \in {}_S\mathbb{M}(\mathcal{S}, \mathcal{S})$ takes the following form. The composite

$$\sigma_X: \phi(X, Y) \xrightarrow{X} \phi(XX, XY) \xrightarrow{\langle \Delta, 1 \rangle^*} \phi(X, XY) \xrightarrow{\langle p, 1 \rangle^*} \phi(1, XY)$$

is an isomorphism and its inverse is given by

$$\sigma_X^{-1}: \phi(1, XY) \xrightarrow{X} \phi(X, XXY) \xrightarrow{\langle 1, \Delta Y \rangle^*} \phi(X, XY) \xrightarrow{\langle 1, pY \rangle^*} \phi(X, Y).$$

6. categories \mathcal{K} and $\mathbb{N}(\mathcal{K})$

The category \mathcal{K} is defined as follows. An object is a diagram

$$X \xleftarrow{a} U \xrightarrow{b} Y$$

of G -sets such that $(a, b): U \rightarrow X \times Y$ is injective. A morphism

$$(X \xleftarrow{a} U \xrightarrow{b} Y) \rightarrow (X' \xleftarrow{a'} U' \xrightarrow{b'} Y')$$

is a triple (f, h, g) of G -maps $f: X \rightarrow X'$, $h: U \rightarrow U'$, $g: Y \rightarrow Y'$ satisfying the obvious commutativity.

\mathcal{K} has finite limits, taken componentwise. In particular we can speak about pull-back diagrams in \mathcal{K} .

The category $\mathbb{N}(\mathcal{K})$ is defined as follows. An object θ consists of k -modules $\theta(\mathbf{X})$ for all objects \mathbf{X} in \mathcal{K} and linear maps

$$\begin{aligned} \mathbf{f}_* &: \theta(\mathbf{X}) \rightarrow \theta(\mathbf{X}') \\ \mathbf{f}_* &: \theta(\mathbf{X}') \rightarrow \theta(\mathbf{X}) \end{aligned}$$

for all morphisms $\mathbf{f}: \mathbf{X} \rightarrow \mathbf{X}'$ in \mathcal{K} , satisfying the following conditions

(6.1.i) $\theta(\mathbf{X})$ and \mathbf{f}_* form a functor $\mathcal{K} \rightarrow \mathcal{V}$.

(6.1.ii) $\theta(\mathbf{X})$ and \mathbf{f}^* form a functor $\mathcal{K}^{\text{op}} \rightarrow \mathcal{V}$.

(6.1.iii) If

$$\begin{array}{ccc} \mathbf{X}_1 & \xrightarrow{\mathbf{f}_1} & \mathbf{X}'_1 \\ \mathbf{p} \downarrow & & \downarrow \mathbf{p}' \\ \mathbf{X}_2 & \xrightarrow{\mathbf{f}_2} & \mathbf{X}'_2 \end{array}$$

is a pull-back diagram in \mathcal{K} , then

$$\begin{array}{ccc} \theta(\mathbf{X}_1) & \xrightarrow{\mathbf{f}_{1*}} & \theta(\mathbf{X}'_1) \\ \mathbf{p}^* \uparrow & & \uparrow \mathbf{p}'^* \\ \theta(\mathbf{X}_2) & \xrightarrow{\mathbf{f}_{2*}} & \theta(\mathbf{X}'_2) \end{array}$$

is commutative.

(6.1.iv) Suppose $\mathbf{X} = (X \leftarrow U_1 + U_2 \rightarrow Y)$ is an object of \mathcal{K} . Let $\mathbf{X}_1 = (X \leftarrow U_1 \rightarrow Y)$, $\mathbf{X}_2 = (X \leftarrow U_2 \rightarrow Y)$ and $\mathbf{i}_1: \mathbf{X}_1 \rightarrow \mathbf{X}$, $\mathbf{i}_2: \mathbf{X}_2 \rightarrow \mathbf{X}$ the obvious injections. Then

$$\begin{aligned} (\mathbf{i}_1^*, \mathbf{i}_2^*) &: \theta(\mathbf{X}) \rightarrow \theta(\mathbf{X}_1) \oplus \theta(\mathbf{X}_2) \\ (\mathbf{i}_{1*}, \mathbf{i}_{2*}) &: \theta(\mathbf{X}_1) \oplus \theta(\mathbf{X}_2) \rightarrow \theta(\mathbf{X}) \end{aligned}$$

are inverse to each other.

(6.1.v) $\theta(X \leftarrow \emptyset \rightarrow Y) = 0$

(6.1.vi) Let $\mathbf{X}_1 = (X_1 \leftarrow U_1 \rightarrow Y)$, $\mathbf{X}_2 = (X_2 \leftarrow U_2 \rightarrow Y)$ be objects in \mathcal{K} and put $\mathbf{X} = (X_1 + X_2 \leftarrow U_1 + U_2 \rightarrow Y)$, $\mathbf{j}_1: \mathbf{X}_1 \rightarrow \mathbf{X}$, $\mathbf{j}_2: \mathbf{X}_2 \rightarrow \mathbf{X}$ the obvious injections. Then

$$\begin{aligned} (\mathbf{j}_1^*, \mathbf{j}_2^*) &: \theta(\mathbf{X}) \rightarrow \theta(\mathbf{X}_1) \oplus \theta(\mathbf{X}_2) \\ (\mathbf{j}_{1*}, \mathbf{j}_{2*}) &: \theta(\mathbf{X}_1) \oplus \theta(\mathbf{X}_2) \rightarrow \theta(\mathbf{X}) \end{aligned}$$

are inverse to each other.

(6.1.vii) The right-sided analogue of (6.1.vi).

(6.1.viii) Let $\mathbf{X} = (X \xleftarrow{a} U \xrightarrow{b} Y)$ be an object of \mathcal{K} . Put $\mathbf{U} = (U \xleftarrow{1} U \xrightarrow{1} U)$ and $\mathbf{a} = (a, 1, b): \mathbf{U} \rightarrow \mathbf{X}$. Then

$$\begin{aligned} \mathbf{a}_* &: \theta(\mathbf{U}) \rightarrow \theta(\mathbf{X}) \\ \mathbf{a}^* &: \theta(\mathbf{X}) \rightarrow \theta(\mathbf{U}) \end{aligned}$$

are inverse to each other.

A morphism $\theta \rightarrow \theta'$ in $\mathbb{N}(\mathcal{K})$ consists of linear maps $\theta(\mathbf{X}) \rightarrow \theta'(\mathbf{X})$ for all objects \mathbf{X} in \mathcal{K} satisfying the commutativity with \mathbf{f}_* and \mathbf{f}^* for all morphisms \mathbf{f} in \mathcal{K} .

This ends the definition of $\mathbb{N}(\mathcal{K})$.

We have an equivalence

$$\mathbb{N}(\mathcal{K}) \simeq \mathbb{M}(\mathcal{S})$$

by the functor sending $\psi \in \mathbb{M}(\mathcal{S})$ to $\phi \in \mathbb{N}(\mathcal{K})$ defined by

$$\phi(X \leftarrow U \rightarrow Y) = \psi(U).$$

7. idempotent operation $e(X \leftarrow U \rightarrow Y)$ on $\phi(X, Y)$

Let ϕ be an object in ${}_S\mathbb{M}(\mathcal{S}, \mathcal{S})$.

For diagrams

$$Z \xleftarrow{c} U \xrightarrow{a} X, \quad Z \xleftarrow{d} V \xrightarrow{b} Y$$

of G -sets, define the map

$$\{Z \xleftarrow{c} U \xrightarrow{a} X, Z \xleftarrow{d} V \xrightarrow{b} Y\}: \phi(X, Y) \rightarrow \phi(X, Y)$$

to be the composite

$$\phi(X, Y) \xrightarrow{Z} \phi(ZX, ZY) \xrightarrow{\langle (c, a), (d, b) \rangle^*} \phi(U, V) \xrightarrow{(a, b)^*} \phi(X, Y).$$

Properties of this map are given below.

(7.1)

$$\begin{array}{ccc} \phi(X, Y) & \xrightarrow{(p_2, p_2)^*} & \phi(ZX, ZY) \\ Z \downarrow & \swarrow & \\ & \{Z \xleftarrow{p_1} ZX \xrightarrow{1} ZX, Z \xleftarrow{p_1} ZY \xrightarrow{1} ZY\} & \\ & \phi(ZX, ZY) & \end{array}$$

where p_1, p_2 are the projections.

Proof.

$$\begin{array}{ccc} \phi(X, Y) & \xrightarrow{\langle p_2, p_2 \rangle^*} & \phi(ZX, ZY) \\ Z \cdot \downarrow & & \downarrow Z. \\ \phi(ZX, ZY) & \xrightarrow{\langle p_{13}, p_{13} \rangle^*} & \phi(ZZX, ZZY) \end{array}$$

Let $\Delta: Z \rightarrow ZX$ denote the diagonal map. Then the composite

$$\phi(ZX, ZY) \xrightarrow{\langle p_{13}, p_{13} \rangle^*} \phi(ZZX, ZZY) \xrightarrow{\langle \Delta X, \Delta Y \rangle^*} \phi(ZX, ZY)$$

is the identity, and the composite

$$\phi(ZX, ZY) \xrightarrow{Z \cdot} \phi(ZZX, ZZY) \xrightarrow{\langle \Delta X, \Delta Y \rangle^*} \phi(ZX, ZY)$$

is

$$\{Z \xleftarrow{p_1} ZX \xrightarrow{1} ZX, Z \xleftarrow{p_1} ZY \xrightarrow{1} ZY\}$$

as $(p_1, 1_{ZX}) = \Delta X$. Hence the conclusion follows.

(7.2) Let

$$\begin{array}{ccccc} Z' & \xleftarrow{c'} & U' & & Z' \\ e \downarrow & & \downarrow f & \searrow a' & e \downarrow & \nwarrow d' \\ Z & \xleftarrow{c} & U & \xrightarrow{a} & X & & Z & \xleftarrow{d} & V & \xrightarrow{b} & Y \end{array}$$

be commutative diagrams in which the left square is pull-back. Then

$$\{Z \xleftarrow{c} U \xrightarrow{a} X, Z \xleftarrow{d} V \xrightarrow{b} Y\} = \{Z' \xleftarrow{c'} U' \xrightarrow{a'} X, Z' \xleftarrow{d'} V \xrightarrow{b} Y\}$$

Proof. We have the commutative diagram

$$\begin{array}{ccccc} \phi(X, Y) & \xrightarrow{Z \cdot} & \phi(ZX, ZY) & & \\ Z' \cdot \downarrow & (a) & \downarrow \langle 1, eY \rangle^* & \searrow \langle 1, (d, b) \rangle^* & \\ \phi(Z'X, Z'Y) & \xrightarrow{\langle eX, 1 \rangle^*} & \phi(ZX, Z'Y) & \xrightarrow{\langle 1, (d', b) \rangle^*} & \phi(ZX, V) \\ \langle (c', a'), 1 \rangle^* \downarrow & (b) & \downarrow \langle (c, a), 1 \rangle^* & & \downarrow \langle (c, a), 1 \rangle^* \\ \phi(U', Z'Y) & \xrightarrow{\langle f, 1 \rangle^*} & \phi(U, Z'Y) & \xrightarrow{\langle 1, (d', b) \rangle^*} & \phi(U, V) \\ & & \langle 1, (d', b) \rangle^* \searrow & (c) & \langle f, 1 \rangle^* \nearrow & \downarrow \langle a, b \rangle^* \\ & & & \phi(U', V) & \xrightarrow{\langle a', b \rangle^*} & \phi(X, Y) \end{array}$$

(a) is commutative by (4.2.ii). (b) is commutative because the diagram

$$\begin{array}{ccc} U' & \xrightarrow{f} & U \\ (c', a') \downarrow & & \downarrow (c, a) \\ Z'X & \xrightarrow{eX} & ZX \end{array}$$

is pull-back. (c) is commutative by (4.1.iii).

The composition of the upper path from $\phi(X, Y)$ to $\phi(X, Y)$ yields $\{Z \leftarrow U \rightarrow X, Z \leftarrow V \rightarrow Y\}$, while the composition of the lower path yields $\{Z' \leftarrow U' \rightarrow X, Z' \leftarrow V \rightarrow Y\}$. Hence the conclusion follows.

(7.3)

$$\begin{array}{ccc} Z' & & Z' \xleftarrow{d'} V' \\ e \downarrow \quad \swarrow c' & & e \downarrow \quad \downarrow g \searrow b' \\ Z \xleftarrow{c} U \xrightarrow{a} X & & Z \xleftarrow{d} V \xrightarrow{b} Y \end{array}$$

the square in the right diagram is pull-back. Then

$$\{Z \xleftarrow{c} U \xrightarrow{a} X, Z \xleftarrow{d} V \xrightarrow{b} Y\} = \{Z' \xleftarrow{c'} U \xrightarrow{a} X, Z' \xleftarrow{d'} V' \xrightarrow{b'} Y\}.$$

(7.4) Let

$$\begin{array}{ccc} W & \xrightarrow{e} & U \\ f \downarrow & & \downarrow c \\ V & \xrightarrow{b} & Z \end{array}$$

be a pull-back diagram. Then

$$\begin{aligned} \{Z \xleftarrow{c} U \xrightarrow{a} X, Z \xleftarrow{d} V \xrightarrow{b} Y\} &= \{X \xleftarrow{1} X \xrightarrow{1} X, X \xleftarrow{ae} W \xrightarrow{bf} Y\} \\ &= \{Y \xleftarrow{bf} W \xrightarrow{ae} X, Y \xleftarrow{1} Y \xrightarrow{1} Y\}. \end{aligned}$$

Proof.

$$\begin{aligned} \{Z \xleftarrow{c} U \xrightarrow{a} X, Z \xleftarrow{d} V \xrightarrow{b} Y\} &= \{U \xleftarrow{1} U \xrightarrow{a} X, U \xleftarrow{e} W \xrightarrow{bf} Y\} \quad \text{by (7.3)} \\ &= \{X \xleftarrow{1} X \xrightarrow{1} X, X \xleftarrow{ae} W \xrightarrow{bf} Y\} \quad \text{by (7.2)} \end{aligned}$$

The second equality is proved similarly.

(7.5) Suppose the diagrams

$$\begin{array}{ccc} & Z & \\ g \nearrow \quad \uparrow c & & h \nearrow \quad \uparrow d \\ U'' \xrightarrow{e} U & & V'' \xrightarrow{f} V \\ g' \swarrow \quad \downarrow e' \searrow a'' \downarrow a & & h' \swarrow \quad \downarrow f' \searrow b'' \downarrow b \\ Z' \xleftarrow{c'} U' \xrightarrow{a'} X & & Z' \xleftarrow{d'} V' \xrightarrow{b'} Y \end{array}$$

are commutative and the squares are pull-back. Then

$$\begin{aligned} \{Z' \xleftarrow{c'} U' \xrightarrow{a'} X, Z' \xleftarrow{d'} V' \xrightarrow{b'} Y\} \circ \{Z \xleftarrow{c} U \xrightarrow{a} X, Z \xleftarrow{d} V \xrightarrow{b} Y\} \\ = \{Z' Z \xleftarrow{(g', g)} U'' \xrightarrow{a''} X, Z' Z \xleftarrow{(h', h)} V'' \xrightarrow{b''} Y\}. \end{aligned}$$

Proof. We will show the following diagram is commutative.

$$\begin{array}{ccccccc}
\phi(X, Y) & \xrightarrow{Z} & \phi(ZX, ZY) & \xrightarrow{\langle (c, a), (d, b) \rangle^*} & \phi(U, V) & \xrightarrow{\langle a, b \rangle^*} & \phi(X, Y) \\
(Z'Z)_* \searrow & & \downarrow Z' & & \downarrow Z' & & \downarrow Z' \\
& & \phi(Z'ZX, Z'ZY) & \xrightarrow{\langle Z'(c, a), Z'(d, b) \rangle^*} & \phi(Z'U, Z'V) & \xrightarrow{\langle Z'a, Z'b \rangle_*} & \phi(Z'X, Z'Y) \\
& & \searrow \langle (g', g, a''), (h', h, b'') \rangle^* & & \text{(I)} & & \downarrow \langle (c', a'), (d', b') \rangle^* \\
& & & & \phi(U'', V'') & & \phi(U', V') \\
& & & & \searrow \langle a'', b'' \rangle_* & & \downarrow \langle a', b' \rangle_* \\
& & & & & & \phi(X, Y)
\end{array}$$

The composition of the horizontal path is

$$\{Z \xleftarrow{c} U \xrightarrow{a} X, Z \xleftarrow{d} V \xrightarrow{b} Y\},$$

the composition of the vertical path is

$$\{Z' \xleftarrow{c'} U' \xrightarrow{a'} X, Z' \xleftarrow{d'} V' \xrightarrow{b'} Y\},$$

and the composition of the oblique path is

$$\{Z'Z \xleftarrow{(g', g)} U'' \xrightarrow{a''} X, Z'Z \xleftarrow{(h', h)} V'' \xrightarrow{b''} Y\}$$

Hence the conclusion will follow from the commutativity.

The small triangle and the two squares in the diagram are commutative by (4.2.iii) and (4.2.i). It remains to show the commutativity of (I). We have the commutative diagram

$$\begin{array}{ccccc}
\phi(Z'ZX, -) & \xrightarrow{\langle Z'(c, a), 1 \rangle^*} & \phi(Z'U, -) & \xrightarrow{\langle Z'a, 1 \rangle^*} & \phi(Z'X, -) \\
\searrow \langle (g', g, a''), 1 \rangle^* & & \downarrow \langle (g', e), 1 \rangle^* \text{(II)} & & \downarrow \langle (c', a'), 1 \rangle^* \\
& & \phi(U'', -) & \xrightarrow{\langle e', 1 \rangle^*} & \phi(U', -) \\
& & \searrow \langle a'', 1 \rangle_* & & \downarrow \langle a', 1 \rangle_* \\
& & & & \phi(X, -)
\end{array}$$

The commutativity of (II) follows from the pull-back diagram

$$\begin{array}{ccc}
U'' & \xrightarrow{(g', e)} & Z'U \\
e' \downarrow & & \downarrow Z'a \\
U' & \xrightarrow{(c', a')} & Z'X
\end{array}$$

We have a similar commutative diagram

$$\begin{array}{ccccc}
\phi(-, Z'ZY) & \xrightarrow{\langle 1, Z'(d,b) \rangle^*} & \phi(-, Z'V) & \xrightarrow{\langle 1, Z'b \rangle^*} & \phi(-, Z'Y) \\
\downarrow \langle 1, (h', h, b'') \rangle^* \searrow & & \downarrow \langle 1, (h', f) \rangle^* & & \downarrow \langle 1, (d', b) \rangle^* \\
& & \phi(-, V'') & \xrightarrow{f'} & \phi(-, V') \\
& & \downarrow \langle 1, b'' \rangle^* \searrow & & \downarrow \langle 1, b' \rangle^* \\
& & & & \phi(-, Y)
\end{array}$$

Combing the two diagrams, we know the commutativity of (I).

(7.6) Suppose the diagram

$$\begin{array}{ccccc}
& & U' & \xrightarrow{a'} & X' \\
& c' \swarrow & \downarrow g & & \downarrow f \\
Z & \xleftarrow{c} & U & \xrightarrow{a} & X
\end{array}$$

is commutative, where the square is pull-back. Suppose

$$Z \xleftarrow{d} V \xrightarrow{b} Y$$

is given. Then the following diagrams are commutative.

$$\begin{array}{ccc}
\phi(X', Y) & \xrightarrow{\{Z \leftarrow U' \rightarrow X', Z \leftarrow V \rightarrow Y\}} & \phi(X', Y) \\
\downarrow \langle f, 1 \rangle^* & & \downarrow \langle f, 1 \rangle^* \\
\phi(X, Y) & \xrightarrow{\{Z \leftarrow U \rightarrow X, Z \leftarrow V \rightarrow Y\}} & \phi(X, Y)
\end{array}$$

and

$$\begin{array}{ccc}
\phi(X', Y) & \xrightarrow{\{Z \leftarrow U' \rightarrow X', Z \leftarrow V \rightarrow Y\}} & \phi(X', Y) \\
\uparrow \langle f, 1 \rangle^* & & \uparrow \langle f, 1 \rangle^* \\
\phi(X, Y) & \xrightarrow{\{Z \leftarrow U \rightarrow X, Z \leftarrow V \rightarrow Y\}} & \phi(X, Y)
\end{array}$$

Proof. The first one follows from the commutative diagram

$$\begin{array}{ccccccc}
\phi(X', Y) & \xrightarrow{Z} & \phi(ZX', ZY) & \xrightarrow{\langle (c', a'), (d, b) \rangle^*} & \phi(U', V) & \xrightarrow{\langle a', b \rangle^*} & \phi(X', Y) \\
\downarrow \langle f, 1 \rangle^* & & \downarrow \langle Zf, 1 \rangle^* & & \downarrow \langle g, 1 \rangle^* & & \downarrow \langle f, 1 \rangle^* \\
\phi(X, Y) & \xrightarrow{Z} & \phi(ZX, ZY) & \xrightarrow{\langle (c, a), (d, b) \rangle^*} & \phi(U, V) & \xrightarrow{\langle a, b \rangle^*} & \phi(X, Y)
\end{array}$$

where commutativity of the middle square is assured by the pull-back

$$\begin{array}{ccc} ZX' & \xleftarrow{(c', a')} & U' \\ Zf \downarrow & & \downarrow g \\ ZX & \xleftarrow{(c, a)} & U \end{array}$$

The second one follows from the commutative diagram

$$\begin{array}{ccccccc} \phi(X', Y) & \xrightarrow{Z.} & \phi(ZX', ZY) & \xrightarrow{\langle (c', a'), (d, b) \rangle^*} & \phi(U', V) & \xrightarrow{\langle a', b \rangle_*} & \phi(X', Y) \\ \uparrow \langle f, 1 \rangle^* & & \uparrow \langle Zf, 1 \rangle^* & & \uparrow \langle g, 1 \rangle^* & & \uparrow \langle f, 1 \rangle^* \\ \phi(X, Y) & \xrightarrow{Z.} & \phi(ZX, ZY) & \xrightarrow{\langle (c, a), (d, b) \rangle^*} & \phi(U, V) & \xrightarrow{\langle a, b \rangle_*} & \phi(X, Y) \end{array}$$

where the commutativity of the right most square is assured by the pull-back

$$\begin{array}{ccc} U' & \xrightarrow{a'} & X' \\ g \downarrow & & \downarrow f \\ U & \xrightarrow{a} & X \end{array}$$

(7.7) The right-sided version of (7.6).

(7.8) For G -maps $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$, we have

$$\begin{aligned} & \langle f, g \rangle_* \circ \{ Z \xleftarrow{c} U \xrightarrow{a} X, Z \xleftarrow{d} V \xrightarrow{b} Y \} \circ \langle f, g \rangle^* \\ &= \{ Z \xleftarrow{c} U \xrightarrow{fa} X', Z \xleftarrow{d} V \xrightarrow{gb} Y' \}. \end{aligned}$$

Proof. This follows from the commutative diagram

$$\begin{array}{ccccccc} \phi(X, Y) & \xrightarrow{Z.} & \phi(ZX, ZY) & \xrightarrow{\langle (c, a), (d, b) \rangle^*} & \phi(U, V) & \xrightarrow{\langle a, b \rangle_*} & \phi(X, Y) \\ \langle f, g \rangle^* \uparrow & & \uparrow \langle Zf, Zg \rangle^* & & \parallel & & \downarrow \langle f, g \rangle_* \\ \phi(X', Y') & \xrightarrow{Z.} & \phi(ZX', ZY') & \xrightarrow{\langle (c, fa), (d, gb) \rangle^*} & \phi(U, V) & \xrightarrow{\langle fa, gb \rangle_*} & \phi(X', Y') \end{array}$$

(7.9) For any diagram

$$X \xleftarrow{a} U \xrightarrow{b} Y$$

the following diagram commutes.

$$\begin{array}{ccc} \phi(X, Y) & \xrightarrow{\sigma_X} & \phi(1, XY) \\ & & \downarrow \langle 1, (a, b) \rangle^* \\ \{ X \leftarrow X \rightarrow X, X \leftarrow U \rightarrow Y \} \downarrow & & \phi(1, U) \\ & & \downarrow \langle 1, (a, b) \rangle_* \\ \phi(X, Y) & \xrightarrow{\sigma_X} & \phi(1, XY) \end{array}$$

Proof. By the commutative diagram

$$\begin{array}{ccccc}
\phi(X, Y) & \xrightarrow{X.} & \phi(XX, XY) & \xrightarrow{\langle \Delta, 1 \rangle^*} & \phi(X, XY) & \xrightarrow{\langle p, 1 \rangle^*} & \phi(1, XY) \\
& & \downarrow \langle 1, (a, b) \rangle^* & & \downarrow \langle 1, (a, b) \rangle^* & & \downarrow \langle 1, (a, b) \rangle^* \\
& & \phi(XX, U) & \xrightarrow{\langle \Delta, 1 \rangle^*} & \phi(X, U) & \xrightarrow{\langle p, 1 \rangle^*} & \phi(1, U) \\
& & \downarrow \langle 1, (a, b) \rangle^* & & \downarrow \langle 1, (a, b) \rangle^* & & \downarrow \langle 1, (a, b) \rangle^* \\
& & \phi(XX, XY) & \xrightarrow{\langle \Delta, 1 \rangle^*} & \phi(X, XY) & \xrightarrow{\langle p, 1 \rangle^*} & \phi(1, XY)
\end{array}$$

$\langle 1, (a, b) \rangle_* \circ \langle 1, (a, b) \rangle^* \circ \sigma_X$ is equal to the composite

$$\begin{array}{c}
\phi(X, Y) \\
\downarrow X. \\
\phi(XX, XY) \\
\downarrow \langle 1, (a, b) \rangle^* \\
\phi(XX, U) \\
\downarrow \langle 1, (a, b) \rangle^* \\
\phi(XX, XY) \xrightarrow{\langle \Delta, 1 \rangle^*} \phi(X, XY) \xrightarrow{\langle p, 1 \rangle^*} \phi(1, XY) \quad (*)
\end{array}$$

On the other hand, by the commutative diagram

$$\begin{array}{ccccc}
\phi(X, Y) & & & & \\
\downarrow X. & \searrow (XX). & & & \\
\phi(XX, XY) & \xrightarrow{X.} & \phi(XXX, XXY) & & \\
\downarrow \langle \Delta, (a, b) \rangle^* & & \downarrow \langle X\Delta, X(a, b) \rangle^* & & \\
\phi(X, U) & \xrightarrow{X.} & \phi(XX, XU) & \xrightarrow{\langle \Delta, 1 \rangle^*} & \phi(X, XU) \\
\downarrow \langle 1, b \rangle_* & & \downarrow \langle 1, Xb \rangle_* & & \downarrow \langle 1, Xb \rangle_* \\
\phi(X, Y) & \xrightarrow{X.} & \phi(XX, XY) & \xrightarrow{\langle \Delta, 1 \rangle^*} & \phi(X, XY) \xrightarrow{\langle p, 1 \rangle^*} \phi(1, XY)
\end{array}$$

and $X\Delta \circ \Delta = \Delta X \circ \Delta$, we see that $\sigma_X \circ \{X \leftarrow X \rightarrow X, X \leftarrow U \rightarrow Y\}$ is equal to the composite

$$\begin{array}{c}
\phi(X, Y) \\
\downarrow (XX). \\
\phi(XXX, XXY) \\
\downarrow \langle \Delta X, X(a, b) \rangle^* \\
\phi(XX, XU) \\
\downarrow \langle 1, Xb \rangle_* \\
\phi(XX, XY) \xrightarrow{\langle \Delta, 1 \rangle^*} \phi(X, XY) \xrightarrow{\langle p, 1 \rangle^*} \phi(1, XY)
\end{array}$$

Moreover

$$\begin{array}{ccc}
\phi(X, Y) & \xrightarrow{\langle XX \rangle_*} & \phi(XXX, XXY) \\
X \downarrow & \text{(I)} & \downarrow \langle \Delta X, 1 \rangle_* \\
\phi(XX, XY) & \xrightarrow{\langle 1, \Delta Y \rangle_*} & \phi(XX, XXY) \\
\langle 1, (a, b) \rangle_* \downarrow & \text{(II)} & \downarrow \langle 1, X(a, b) \rangle_* \\
\phi(XX, U) & \xrightarrow{\langle 1, (a, 1) \rangle_*} & \phi(XX, XU) \\
& \searrow \langle 1, (a, b) \rangle_* & \downarrow \langle 1, Xb \rangle_* \\
& & \phi(XX, XY)
\end{array}$$

Here the commutativity of (I) is assured by (4.2.ii) and the commutativity of (II) by the pull-back

$$\begin{array}{ccc}
U & \xrightarrow{(a, 1)} & XU \\
(a, b) \downarrow & & \downarrow \Delta Y \\
XY & \xrightarrow{X(a, b)} & XXY
\end{array}$$

Hence $\sigma_X \circ \{X \leftarrow X \rightarrow X, X \leftarrow U \rightarrow Y\}$ is equal to the composite (*) as well. This proves (7.9).

If $(a, b): U \rightarrow X \times Y$ is injective, put

$$\begin{aligned}
e(X \xleftarrow{a} U \xrightarrow{b} Y) &= \{X \xleftarrow{1} X \xrightarrow{1} X, X \xleftarrow{a} U \xrightarrow{b} Y\} \\
&= \{Y \xleftarrow{b} U \xrightarrow{a} X, Y \xleftarrow{1} Y \xrightarrow{1} Y\}
\end{aligned}$$

which is an endomorphism of $\phi(X, Y)$.

(7.10) Let $f: X' \rightarrow X, g: Y' \rightarrow Y$ be G -maps. Let

$$\begin{array}{ccc}
U' & \xrightarrow{(a', b')} & X' \times Y' \\
h \downarrow & & \downarrow f \times g \\
U & \xrightarrow{(a, b)} & X \times Y
\end{array}$$

be a pull-back diagram. Then the following diagrams are commutative.

$$\begin{array}{ccc}
\phi(X', Y') & \xrightarrow{e(X' \xleftarrow{a'} U \xrightarrow{b'} Y')} & \phi(X', Y') \\
\langle f, g \rangle_* \downarrow & & \downarrow \langle f, g \rangle_* \\
\phi(X, Y) & \xrightarrow{e(X \xleftarrow{a} U \xrightarrow{b} Y)} & \phi(X, Y)
\end{array}$$

$$\begin{array}{ccc}
\phi(X', Y') & \xrightarrow{e(X' \xleftarrow{a'} U \xrightarrow{b'} Y')} & \phi(X', Y') \\
\langle f, g \rangle^* \uparrow & & \uparrow \langle f, g \rangle^* \\
\phi(X, Y) & \xrightarrow{e(X \xleftarrow{a} U \xrightarrow{b} Y)} & \phi(X, Y)
\end{array}$$

Proof. It is enough to show the commutativity in the case where $f = 1$ and the case where $g = 1$. We will consider only the latter case.

By (7.7)

$$\begin{array}{ccc}
\phi(X', Y) & \xrightarrow{\{X \leftarrow X' \rightarrow X', X \leftarrow U \rightarrow Y\}} & \phi(X', Y) \\
\langle f, 1 \rangle_* \downarrow & & \downarrow \langle f, 1 \rangle_* \\
\phi(X, Y) & \xrightarrow{\{X \leftarrow X \rightarrow X, X \leftarrow U \rightarrow Y\}} & \phi(X, Y)
\end{array}$$

By (7.3)

$$\{X \leftarrow X' \rightarrow X', X \leftarrow U \rightarrow Y\} = \{X' \leftarrow X' \rightarrow X', X' \leftarrow U' \rightarrow Y\}.$$

The proof for $\langle f, g \rangle^*$ is similar.

(7.11)

$$e(X \xleftarrow{p_1} XY \xrightarrow{p_2} Y) = 1.$$

Proof. This is clear from (7.9). Or

$$\begin{aligned}
\{X \leftarrow X \rightarrow X, X \xleftarrow{p_1} XY \xrightarrow{p_2} Y\} &= \{1 \leftarrow X \rightarrow X, 1 \leftarrow Y \rightarrow Y\} \quad \text{by (7.3)} \\
&= 1.
\end{aligned}$$

(7.12)

$$\begin{array}{ccc}
U'' & \longrightarrow & U' \\
\downarrow & & \downarrow \\
U & \longrightarrow & XY
\end{array}$$

be a pull-back. Then

$$e(X \leftarrow U \rightarrow Y) \circ e(X \leftarrow U' \rightarrow Y) = e(X \leftarrow U'' \rightarrow Y).$$

Proof. This is clear from (7.9). Or, let

$$\begin{array}{ccc}
W & \longrightarrow & U' \\
\downarrow & & \downarrow \\
U & \longrightarrow & Y
\end{array}$$

be a pull-back. Then we have a pull-back diagram

$$\begin{array}{ccc}
U'' & \longrightarrow & W \\
\downarrow & & \downarrow \\
X & \xrightarrow{\Delta} & XX
\end{array}$$

Hence

$$\begin{aligned}
& \{X \leftarrow X \rightarrow X, X \leftarrow U \rightarrow Y\} \circ \{X \leftarrow X \rightarrow X, X \leftarrow U' \rightarrow Y\} \\
&= \{XX \xleftarrow{\Delta} X \rightarrow X, XX \leftarrow W \rightarrow Y\} \quad \text{by (7.5)} \\
&= \{X \leftarrow X \rightarrow X, X \leftarrow U'' \rightarrow Y\} \quad \text{by (7.3)}
\end{aligned}$$

(7.13)

$$e(X \leftarrow U_1 + U_2 \rightarrow Y) = e(X \leftarrow U_1 \rightarrow Y) + e(X \leftarrow U_2 \rightarrow Y).$$

Proof. This is clear from the definition.

8. equivalence ${}_S\mathbb{M}(\mathcal{S}, \mathcal{S}) \simeq \mathbb{N}(\mathcal{K})$

We already know by Theorem 2.3 that ${}_S\mathbb{M}(\mathcal{S}, \mathcal{S}) \simeq \mathbb{M}(\mathcal{S})$, and hence ${}_S\mathbb{M}(\mathcal{S}, \mathcal{S}) \simeq \mathbb{N}(\mathcal{K})$. In this section we construct the equivalence in another way.

Suppose given $\phi \in {}_S\mathbb{M}(\mathcal{S}, \mathcal{S})$. We will construct an object $\theta \in \mathbb{N}(\mathcal{K})$.

For an object $\mathbf{X} = (X \leftarrow U \rightarrow Y)$ of \mathcal{K} , we have the idempotent

$$e(\mathbf{X}): \phi(X, Y) \rightarrow \phi(X, Y).$$

Let

$$\theta(\mathbf{X}) = \text{Im } e(\mathbf{X})$$

and

$$\text{in}_{\mathbf{X}}: \theta(\mathbf{X}) \rightarrow \phi(X, Y), \quad \text{pr}_{\mathbf{X}}: \phi(X, Y) \rightarrow \theta(\mathbf{X})$$

the inclusion and the projection.

Let $\mathbf{f}: \mathbf{X} \rightarrow \mathbf{X}'$ be a morphism in \mathcal{K} with $\mathbf{X} = (X \leftarrow U \rightarrow Y)$, $\mathbf{X}' = (X' \leftarrow U' \rightarrow Y')$, $\mathbf{f} = (f, h, g)$. Define the map \mathbf{f}_* by

$$\begin{array}{ccc}
\phi(X, Y) & \xleftarrow{\text{in}_{\mathbf{X}}} & \theta(\mathbf{X}) \\
\langle f, g \rangle_* \downarrow & & \downarrow \mathbf{f}_* \\
\phi(X', Y') & \xrightarrow{\text{pr}_{\mathbf{X}'}} & \theta(\mathbf{X}')
\end{array}$$

and \mathbf{f}^* by

$$\begin{array}{ccc}
\phi(X, Y) & \xrightarrow{\text{pr}_{\mathbf{X}}} & \theta(\mathbf{X}) \\
\langle f, g \rangle_* \uparrow & & \uparrow \mathbf{f}^* \\
\phi(X', Y') & \xleftarrow{\text{in}_{\mathbf{X}'}} & \theta(\mathbf{X}')
\end{array}$$

We will show that $\theta(\mathbf{X})$, \mathbf{f}_* , \mathbf{f}^* satisfy the conditions of Section 6 so that θ is an object of $\mathbb{N}(\mathcal{K})$.

Lemma 8.1. Let $\mathbf{X} = (X \leftarrow U \rightarrow Y)$ and $\mathbf{X}' = (X' \leftarrow U' \rightarrow Y')$. Let $f: X \rightarrow X'$, $g: Y \rightarrow Y'$ be G -maps. Suppose $U \rightarrow XY$ and $U' \rightarrow X'Y'$ are the inclusion maps and $U \cap (f \times g)^{-1}(U') = \emptyset$. Then

$$\text{pr}_{\mathbf{X}'} \circ \langle f, g \rangle_* \circ \text{in}_{\mathbf{X}} = 0$$

$$\text{pr}_{\mathbf{X}} \circ \langle f, g \rangle^* \circ \text{in}_{\mathbf{X}'} = 0.$$

Proof. Put $U_1 = (f \times g)^{-1}(U')$ and put $\mathbf{X}_1 = (X \leftarrow U_1 \rightarrow Y)$. Then

$$e_{\mathbf{X}} \circ e_{\mathbf{X}_1} = 0.$$

Also by (7.10)

$$\langle f, g \rangle_* \circ e(\mathbf{X}_1) = e(\mathbf{X}') \circ \langle f, g \rangle_*$$

$$e(\mathbf{X}_1) \circ \langle f, g \rangle^* = \langle f, g \rangle^* \circ e(\mathbf{X}').$$

Hence

$$e_{\mathbf{X}'} \circ \langle f, g \rangle_* \circ e_{\mathbf{X}} = \langle f, g \rangle_* \circ e(\mathbf{X}_1) \circ e_{\mathbf{X}} = 0$$

and

$$e_{\mathbf{X}} \circ \langle f, g \rangle^* \circ e_{\mathbf{X}'} = e_{\mathbf{X}} \circ e(\mathbf{X}_1) \circ \langle f, g \rangle^* = 0.$$

(6.1.i, ii) For morphisms $\mathbf{f}: \mathbf{X} \rightarrow \mathbf{X}'$ and $\mathbf{f}': \mathbf{X}' \rightarrow \mathbf{X}''$,

$$(\mathbf{f}' \circ \mathbf{f})_* = \mathbf{f}'_* \circ \mathbf{f}_*$$

$$(\mathbf{f}' \circ \mathbf{f})^* = \mathbf{f}^* \circ \mathbf{f}'^*$$

Proof. Write

$$\mathbf{X} = (X \leftarrow U \rightarrow Y), \quad \mathbf{X}' = (X' \leftarrow U' \rightarrow Y'), \quad \mathbf{X}'' = (X \leftarrow U'' \rightarrow Y'')$$

and

$$\mathbf{f} = (f, h, g), \quad \mathbf{f}' = (f', h', g').$$

We may assume that $U \rightarrow XY$ and $U' \rightarrow X'Y'$ are the inclusion maps. Let U'_1 be the complement of U' in $X' \times Y'$ and put $\mathbf{X}'_1 = (X' \leftarrow U'_1 \rightarrow Y')$. Since $X'Y' = U' + U'_1$, we have $1 = e(\mathbf{X}') + e(\mathbf{X}'_1)$. Since $U \cap (f \times g)^{-1}(U'_1) = \emptyset$, we have by lemma

$$\text{pr}_{\mathbf{X}'_1} \circ \langle f, g \rangle_* \circ \text{in}_{\mathbf{X}} = 0 \tag{1}$$

$$\text{pr}_{\mathbf{X}} \circ \langle f, g \rangle^* \circ \text{in}_{\mathbf{X}'_1} = 0 \tag{2}$$

$$\langle f' \circ f, h' \circ h, g' \circ g \rangle_*$$

$$= \text{pr}_{\mathbf{X}''} \circ \langle f' \circ f, g' \circ g \rangle_* \circ \text{in}_{\mathbf{X}}$$

$$= \text{pr}_{\mathbf{X}''} \circ \langle f', g' \rangle_* \circ \langle f, g \rangle_* \circ \text{in}_{\mathbf{X}}$$

$$= \text{pr}_{\mathbf{X}''} \circ \langle f', g' \rangle_* \circ (e_{\mathbf{X}'} + e_{\mathbf{X}'_1}) \circ \langle f, g \rangle_* \circ \text{in}_{\mathbf{X}}$$

$$= \text{pr}_{\mathbf{X}''} \circ \langle f', g' \rangle_* \circ e_{\mathbf{X}'} \circ \langle f, g \rangle_* \circ \text{in}_{\mathbf{X}} + \text{pr}_{\mathbf{X}''} \circ \langle f', g' \rangle_* \circ e_{\mathbf{X}'_1} \circ \langle f, g \rangle_* \circ \text{in}_{\mathbf{X}}$$

The second term vanishes by (1) and the first term equals $\langle f', h', g' \rangle_* \circ \langle f, h, g \rangle_*$. This proves the identity $(\mathbf{f}' \circ \mathbf{f})_* = \mathbf{f}'_* \circ \mathbf{f}_*$.

$$\begin{aligned}
& \langle f' \circ f, h' \circ h, g' \circ g \rangle_* \\
&= \text{pr}_{\mathbf{X}} \circ \langle f' \circ f, g' \circ g \rangle^* \circ \text{in}_{\mathbf{X}''} \\
&= \text{pr}_{\mathbf{X}} \circ \langle f, g \rangle^* \circ \langle f', g' \rangle^* \circ \text{in}_{\mathbf{X}''} \\
&= \text{pr}_{\mathbf{X}} \circ \langle f, g \rangle^* \circ (e_{\mathbf{X}'} + e_{\mathbf{X}'_1}) \circ \langle f', g' \rangle^* \circ \text{in}_{\mathbf{X}''} \\
&= \text{pr}_{\mathbf{X}} \circ \langle f, g \rangle^* \circ e_{\mathbf{X}'} \circ \langle f', g' \rangle^* \circ \text{in}_{\mathbf{X}''} + \text{pr}_{\mathbf{X}} \circ \langle f, g \rangle^* \circ e_{\mathbf{X}'_1} \circ \langle f', g' \rangle^* \circ \text{in}_{\mathbf{X}''}
\end{aligned}$$

The second term vanishes by (2). This prove the identity $(\mathbf{f}' \circ \mathbf{f})^* = \mathbf{f}^* \circ \mathbf{f}'^*$.

(6.1.iii) If

$$\begin{array}{ccc}
\mathbf{X}_1 & \xrightarrow{\mathbf{f}_1} & \mathbf{X}'_1 \\
\mathbf{p} \downarrow & & \downarrow \mathbf{p}' \\
\mathbf{X}_2 & \xrightarrow{\mathbf{f}_2} & \mathbf{X}'_2
\end{array}$$

is a pull-back diagram in \mathcal{K} , then

$$\begin{array}{ccc}
\theta(\mathbf{X}_1) & \xrightarrow{\mathbf{f}_{1*}} & \theta(\mathbf{X}'_1) \\
\mathbf{p}^* \uparrow & & \uparrow \mathbf{p}'^* \\
\theta(\mathbf{X}_2) & \xrightarrow{\mathbf{f}_{2*}} & \theta(\mathbf{X}'_2)
\end{array}$$

is commutative.

Proof. Write

$$\mathbf{X}_i = (X_i \leftarrow U_i \rightarrow Y_i), \quad \mathbf{X}'_i = (X'_i \leftarrow U'_i \rightarrow Y'_i)$$

for $i = 1, 2$ and

$$\mathbf{f}_i = (f_i, h_i, g_i), \quad \mathbf{p} = (p, r, q), \quad \mathbf{p}' = (p', r', q')$$

We may assume $U_i \rightarrow X_i Y_i$, $U'_i \rightarrow X'_i Y'_i$ are the inclusion maps. Put

$$\begin{aligned}
\mathbf{V}_1 &= (X_1 \leftarrow (f_1 \times g_1)^{-1}(U'_1) \rightarrow Y_1), \\
\mathbf{V}_2 &= (X_2 \leftarrow (f_2 \times g_2)^{-1}(U'_2) \rightarrow Y_2), \\
\mathbf{W} &= (X_1 \leftarrow (p \times q)^{-1}(U_2) \rightarrow Y_1)
\end{aligned}$$

By (7.10),

$$e(\mathbf{X}'_1) \circ \langle f_1, g_1 \rangle_* = \langle f_1, g_1 \rangle_* \circ e(\mathbf{V}_1) \quad (1)$$

$$e(\mathbf{X}'_2) \circ \langle f_2, g_2 \rangle_* = \langle f_2, g_2 \rangle_* \circ e(\mathbf{V}_2) \quad (2)$$

$$\langle p, q \rangle^* \circ e(\mathbf{X}_2) = e(\mathbf{W}) \circ \langle p, q \rangle^* \quad (3)$$

Since $(f_1 \times g_1)^{-1}(U'_1) \cap (p \times q)^{-1}(U_2) = U_1$, we have

$$e(\mathbf{V}_1) \circ e(\mathbf{W}) = e(\mathbf{X}_1) \quad (4)$$

Since $U_2 \subset (f_2 \times g_2)^{-1}(U'_2)$, we have

$$e(\mathbf{V}_2) \circ e(\mathbf{X}_2) = e(\mathbf{X}_2) \quad (5)$$

$$\begin{aligned} \mathbf{p}'^* \circ \mathbf{f}_{2*} &= \text{pr}_{\mathbf{X}'_1} \circ \langle p', q' \rangle^* \circ \text{in}_{\mathbf{X}'_2} \circ \text{pr}_{\mathbf{X}'_2} \circ \langle f_2, g_2 \rangle_* \circ \text{in}_{\mathbf{X}_2} \\ &= \text{pr}_{\mathbf{X}'_1} \circ \langle p', q' \rangle^* \circ e(\mathbf{X}'_2) \circ \langle f_2, g_2 \rangle_* \circ \text{in}_{\mathbf{X}_2} \\ &= \text{pr}_{\mathbf{X}'_1} \circ \langle p', q' \rangle^* \circ \langle f_2, g_2 \rangle_* \circ e(\mathbf{V}_2) \circ \text{in}_{\mathbf{X}_2} \quad \text{by (2)} \\ &= \text{pr}_{\mathbf{X}'_1} \circ \langle p', q' \rangle^* \circ \langle f_2, g_2 \rangle_* \circ \text{in}_{\mathbf{X}_2} \quad \text{by (5)} \end{aligned} \quad (6)$$

On the other hand

$$\begin{aligned} \mathbf{f}_{1*} \circ \mathbf{p}^* &= \text{pr}_{\mathbf{X}'_1} \circ \langle f_1, g_1 \rangle_* \circ \text{in}_{\mathbf{X}_1} \circ \text{pr}_{\mathbf{X}_1} \circ \langle p, q \rangle^* \circ \text{in}_{\mathbf{X}_2} \\ &= \text{pr}_{\mathbf{X}'_1} \circ \langle f_1, g_1 \rangle_* \circ e(\mathbf{X}_1) \circ \langle p, q \rangle^* \circ \text{in}_{\mathbf{X}_2} \\ &= \text{pr}_{\mathbf{X}'_1} \circ \langle f_1, g_1 \rangle_* \circ e(\mathbf{V}_1) \circ e(\mathbf{W}) \circ \langle p, q \rangle^* \circ \text{in}_{\mathbf{X}_2} \\ &= \text{pr}_{\mathbf{X}'_1} \circ e(\mathbf{X}'_1) \circ \langle f_1, g_1 \rangle_* \circ \langle p, q \rangle^* \circ e(\mathbf{X}_2) \circ \text{in}_{\mathbf{X}_2} \quad \text{by (1), (3)} \\ &= \text{pr}_{\mathbf{X}'_1} \circ \langle f_1, g_1 \rangle_* \circ \langle p, q \rangle^* \circ \text{in}_{\mathbf{X}_2} \end{aligned} \quad (7)$$

By (4.1.iv), (4.1.v)

$$\langle p', q' \rangle^* \circ \langle f_2, g_2 \rangle_* = \langle f_1, g_1 \rangle_* \circ \langle p, q \rangle^*. \quad (8)$$

It follows from (6), (7), and (8)

$$\mathbf{p}'^* \circ \mathbf{f}_{2*} = \mathbf{f}_{1*} \circ \mathbf{p}^*.$$

(6.1.iv) Suppose $\mathbf{X} = (X \leftarrow U_1 + U_2 \rightarrow Y)$ is an object of \mathcal{K} . Let $\mathbf{X}_1 = (X \leftarrow U_1 \rightarrow Y)$, $\mathbf{X}_2 = (X \leftarrow U_2 \rightarrow Y)$ and $\mathbf{i}_1: \mathbf{X}_1 \rightarrow \mathbf{X}$, $\mathbf{i}_2: \mathbf{X}_2 \rightarrow \mathbf{X}$ the obvious injections. Then

$$(\mathbf{i}_{1*}, \mathbf{i}_{2*}): \theta(\mathbf{X}_1) \oplus \theta(\mathbf{X}_2) \rightarrow \theta(\mathbf{X})$$

is an isomorphism.

Proof. We know by (7.13) that the idempotent $e(\mathbf{X})$ is the sum of the mutually orthogonal idempotents $e(\mathbf{X}_1)$ and $e(\mathbf{X}_2)$, so

$$\theta(\mathbf{X}) = \theta(\mathbf{X}_1) \oplus \theta(\mathbf{X}_2)$$

and the inclusion maps $\theta(\mathbf{X}_i) \rightarrow \theta(\mathbf{X})$ for $i = 1, 2$ coincide with the maps

$$\text{pr}_{\mathbf{X}} \circ \text{in}_{\mathbf{X}_i} = \text{pr}_{\mathbf{X}} \circ \langle 1_X, 1_Y \rangle_* \circ \text{in}_{\mathbf{X}_i} = \mathbf{i}_{i*}.$$

(6.1.v) Let $\mathbf{X}_1 = (X_1 \leftarrow U_1 \rightarrow Y)$, $\mathbf{X}_2 = (X_2 \leftarrow U_2 \rightarrow Y)$ be objects in \mathcal{K} and put $\mathbf{X} = (X_1 + X_2 \leftarrow U_1 + U_2 \rightarrow Y)$, $\mathbf{j}_1: \mathbf{X}_1 \rightarrow \mathbf{X}$, $\mathbf{j}_2: \mathbf{X}_2 \rightarrow \mathbf{X}$ the obvious injections. Then

$$(\mathbf{j}_{1*}, \mathbf{j}_{2*}): \theta(\mathbf{X}_1) \oplus \theta(\mathbf{X}_2) \rightarrow \theta(\mathbf{X})$$

is an isomorphism.

Proof. We know that

$$(\langle j_1, 1 \rangle_*, \langle j_2, 1 \rangle_*): \phi(X_1, Y) \oplus \phi(X_2, Y) \rightarrow \phi(X, Y) \quad (*)$$

is an isomorphism. Since

$$\begin{array}{ccc} U_i & \longrightarrow & X_i Y \\ \downarrow & & \downarrow \\ U_1 + U_2 & \longrightarrow & (X_1 + X_2)Y \end{array}$$

is pull-back, we have

$$\langle j_i, 1 \rangle_* \circ e(\mathbf{X}) = e(\mathbf{X}_i) \circ \langle j_i, 1 \rangle.$$

It follows that the isomorphism $(*)$ restricts to the required isomorphism.

(6.1.viii) Let $\mathbf{X} = (X \xleftarrow{a} U \xrightarrow{b} Y)$ be an object of \mathcal{K} . Put $\mathbf{U} = (U \xleftarrow{1} U \xrightarrow{1} U)$ and $\mathbf{a} = (a, 1, b): \mathbf{U} \rightarrow \mathbf{X}$. Then

$$\mathbf{a}_*: \theta(\mathbf{U}) \rightarrow \theta(\mathbf{X})$$

is an isomorphism.

Proof. By (7.9) we have an isomorphism

$$\tau_{\mathbf{X}}: \theta(\mathbf{X}) \rightarrow \phi(\mathbf{1}, U)$$

so that the diagrams

$$\begin{array}{ccccc} \phi(X, Y) & \xrightarrow{\sigma_X} & \phi(\mathbf{1}, XY) & \phi(X, Y) & \xrightarrow{\sigma_X} & \phi(\mathbf{1}, XY) \\ \text{in}_{\mathbf{X}} \uparrow & & \uparrow \langle 1, (a, b) \rangle_* \text{pr}_{\mathbf{X}} & & \downarrow \langle 1, (a, b) \rangle^* \\ \theta(\mathbf{X}) & \xrightarrow{\tau_{\mathbf{X}}} & \phi(\mathbf{1}, U) & \theta(\mathbf{X}) & \xrightarrow{\tau_{\mathbf{X}}} & \phi(\mathbf{1}, U) \end{array}$$

are commutative. For the object \mathbf{U} we have a similar isomorphism

$$\tau_{\mathbf{U}}: \theta(\mathbf{U}) \rightarrow \phi(\mathbf{1}, U).$$

We know

$$\begin{array}{ccc} \phi(X, Y) & \xrightarrow{\sigma_X} & \phi(\mathbf{1}, XY) \\ \langle a, b \rangle_* \uparrow & & \uparrow \langle 1, a \times b \rangle_* \\ \phi(U, U) & \xrightarrow{\sigma_U} & \phi(\mathbf{1}, UU) \end{array}$$

Hence

$$\begin{array}{ccc}
\theta(\mathbf{X}) & \xrightarrow{\tau_{\mathbf{X}}} & \phi(\mathbf{1}, U) \\
\text{pr}_{\mathbf{X}} \uparrow & & \uparrow \langle 1, (a, b) \rangle^* \\
\phi(X, Y) & & \phi(\mathbf{1}, XY) \\
\langle a, b \rangle_* \uparrow & & \uparrow \langle 1, a \times b \rangle_* \\
\phi(U, U) & & \phi(\mathbf{1}, UU) \\
\text{in}_{\mathbf{U}} \uparrow & & \uparrow \langle 1, \Delta \rangle_* \\
\theta(\mathbf{U}) & \xrightarrow{\tau_{\mathbf{U}}} & \phi(\mathbf{1}, U)
\end{array}$$

But

$$\langle 1, (a, b) \rangle^* \circ \langle 1, a \times b \rangle_* \circ \langle 1, \Delta \rangle_* = \langle 1, (a, b) \rangle^* \circ \langle 1, (a, b) \rangle_* = 1$$

as $(a, b): U \rightarrow XY$ is injective. Hence

$$\begin{array}{ccc}
\theta(\mathbf{X}) & \xrightarrow{\tau_{\mathbf{X}}} & \phi(\mathbf{1}, U) \\
\mathbf{a}_* \uparrow & & \uparrow 1 \\
\theta(\mathbf{U}) & \xrightarrow{\tau_{\mathbf{U}}} & \phi(\mathbf{1}, U)
\end{array}$$

So \mathbf{a}_* is an isomorphism.

9. categories \mathcal{H} and $\mathbb{N}(\mathcal{H})$

The category \mathcal{H} is defined as follows. An object of \mathcal{H} is a diagram

$$\begin{array}{ccc}
& U & \\
\swarrow & & \searrow \\
X & & Y \\
\swarrow & & \searrow \\
& V &
\end{array}$$

of G -sets such that the maps $U \rightarrow X \times Y$ and $V \rightarrow X \times Y$ are injective.

A morphism

$$\left(\begin{array}{ccc} & U & \\ \swarrow & & \searrow \\ X & & Y \\ \swarrow & & \searrow \\ & V & \end{array} \right) \rightarrow \left(\begin{array}{ccc} & U' & \\ \swarrow & & \searrow \\ X' & & Y' \\ \swarrow & & \searrow \\ & V' & \end{array} \right)$$

in \mathcal{H} is a quadruple

$$\begin{pmatrix} h \\ f \ g \\ k \end{pmatrix}$$

of G -maps $f: X \rightarrow X'$, $g: Y \rightarrow Y'$, $h: U \rightarrow U'$, $k: V \rightarrow V'$ making the four squares commutative.

The category \mathcal{H} has finite inverse limits given by componentwise.

The category $\mathbb{N}(\mathcal{H})$ is defined as follows. An object θ consists of k -modules $\theta(\mathbf{X})$ for all objects \mathbf{X} in \mathcal{H} and linear maps

$$\begin{aligned} \mathbf{f}_* &: \theta(\mathbf{X}) \rightarrow \theta(\mathbf{X}') \\ \mathbf{f}_* &: \theta(\mathbf{X}') \rightarrow \theta(\mathbf{X}) \end{aligned}$$

for all morphisms $\mathbf{f}: \mathbf{X} \rightarrow \mathbf{X}'$ in \mathcal{H} , satisfying the following conditions

- (9.1.i) $\theta(\mathbf{X})$ and \mathbf{f}_* form a functor $\mathcal{H} \rightarrow \mathcal{V}$.
- (9.1.ii) $\theta(\mathbf{X})$ and \mathbf{f}^* form a functor $\mathcal{H}^{\text{op}} \rightarrow \mathcal{V}$.
- (9.1.iii) If

$$\begin{array}{ccc} \mathbf{X}_1 & \xrightarrow{\mathbf{f}_1} & \mathbf{X}'_1 \\ \mathbf{p} \downarrow & & \downarrow \mathbf{p}' \\ \mathbf{X}_2 & \xrightarrow{\mathbf{f}_2} & \mathbf{X}'_2 \end{array}$$

is a pull-back diagram in \mathcal{H} , then

$$\begin{array}{ccc} \theta(\mathbf{X}_1) & \xrightarrow{\mathbf{f}_{1*}} & \theta(\mathbf{X}'_1) \\ \mathbf{p}^* \uparrow & & \uparrow \mathbf{p}'^* \\ \theta(\mathbf{X}_2) & \xrightarrow{\mathbf{f}_{2*}} & \theta(\mathbf{X}'_2) \end{array}$$

is commutative.

(9.1.iv) Suppose

$$\mathbf{X} = \left(\begin{array}{ccc} & U_1 + U_2 & \\ \swarrow & & \searrow \\ X & & Y \\ \nwarrow & & \nearrow \\ & V & \end{array} \right)$$

is an object of \mathcal{H} . Put

$$\mathbf{X}_1 = \left(\begin{array}{ccc} & U_1 & \\ \swarrow & & \searrow \\ X & & Y \\ \nwarrow & & \nearrow \\ & V & \end{array} \right), \quad \mathbf{X}_2 = \left(\begin{array}{ccc} & U_2 & \\ \swarrow & & \searrow \\ X & & Y \\ \nwarrow & & \nearrow \\ & V & \end{array} \right)$$

and let $\mathbf{i}_1: \mathbf{X}_1 \rightarrow \mathbf{X}$, $\mathbf{i}_2: \mathbf{X}_2 \rightarrow \mathbf{X}$ the obvious injections. Then

$$\begin{aligned} (\mathbf{i}_1^*, \mathbf{i}_2^*) &: \theta(\mathbf{X}) \rightarrow \theta(\mathbf{X}_1) \oplus \theta(\mathbf{X}_2) \\ (\mathbf{i}_{1*}, \mathbf{i}_{2*}) &: \theta(\mathbf{X}_1) \oplus \theta(\mathbf{X}_2) \rightarrow \theta(\mathbf{X}) \end{aligned}$$

are inverse to each other.

(9.1.v)

$$\theta \left(\begin{array}{ccc} & \emptyset & \\ \swarrow & & \searrow \\ X & & Y \\ \nwarrow & & \nearrow \\ & V & \end{array} \right) = 0$$

- (9.1.vi) The V -version of (9.1.iv).
 (9.1.vii) The V -version of (9.1.iv).
 (9.1.viii) Let

$$\mathbf{X}_1 = \left(\begin{array}{ccc} & U_1 & \\ \swarrow & & \searrow \\ X_1 & & Y \\ \nwarrow & & \nearrow \\ & V_1 & \end{array} \right), \quad \mathbf{X}_2 = \left(\begin{array}{ccc} & U_2 & \\ \swarrow & & \searrow \\ X_2 & & Y \\ \nwarrow & & \nearrow \\ & V_2 & \end{array} \right)$$

be objects in \mathcal{H} . Put

$$\mathbf{X} = \left(\begin{array}{ccc} & U_1 + U_2 & \\ \swarrow & & \searrow \\ X_1 + X_2 & & Y \\ \nwarrow & & \nearrow \\ & V_1 + V_2 & \end{array} \right)$$

and let $\mathbf{j}_1: \mathbf{X}_1 \rightarrow \mathbf{X}$, $\mathbf{j}_2: \mathbf{X}_2 \rightarrow \mathbf{X}$ be the obvious injections. Then

$$\begin{aligned} (\mathbf{j}_1^*, \mathbf{j}_2^*): \theta(\mathbf{X}) &\rightarrow \theta(\mathbf{X}_1) \oplus \theta(\mathbf{X}_2) \\ (\mathbf{j}_{1*}, \mathbf{j}_{2*}): \theta(\mathbf{X}_1) \oplus \theta(\mathbf{X}_2) &\rightarrow \theta(\mathbf{X}) \end{aligned}$$

are inverse to each other.

- (9.1.ix) The right-sided analogue of (9.1.viii).
 (9.1.x) Let

$$\mathbf{X} = \left(\begin{array}{ccc} & U & \\ f \swarrow & & \searrow g \\ X & & Y \\ h \nwarrow & & \nearrow k \\ & V & \end{array} \right)$$

be an object in \mathcal{H} . Let

$$\begin{array}{ccc} V_1 & \xrightarrow{(h_1, k_1)} & U \times U \\ v \downarrow & & \downarrow f \times g \\ V & \xrightarrow{(h, k)} & X \times Y \end{array}$$

be a pull-back. Put

$$\mathbf{U} = \left(\begin{array}{ccc} & U & \\ 1 \swarrow & & \searrow 1 \\ U & & U \\ h_1 \nwarrow & & \nearrow k_1 \\ & V_1 & \end{array} \right)$$

and

$$\mathbf{f} = \begin{pmatrix} 1 \\ f & g \\ v \end{pmatrix}: \mathbf{U} \rightarrow \mathbf{X}$$

a morphism in \mathcal{H} . Then

$$f_*: \theta(U) \rightarrow \theta(X)$$

$$f^*: \theta(X) \rightarrow \theta(U)$$

are inverse to each other.

(9.1.xi) The V -version of (9.1.x).

A morphism $\theta \rightarrow \theta'$ in $\mathbb{N}(\mathcal{H})$ consists of linear maps $\theta(X) \rightarrow \theta'(X)$ for all objects X in \mathcal{H} satisfying the commutativity with f_* and f^* for all morphisms f in \mathcal{H} .

This ends the definition of $\mathbb{N}(\mathcal{H})$.

10. idempotents $e^L(X \leftarrow U \rightarrow Y)$ and $e^R(X \leftarrow V \rightarrow Y)$

Let ϕ be an object of ${}_S\mathbb{M}(\mathcal{S}, \mathcal{S})_S$. As defined in Section 7, the left-sided operations $Z.$ for ϕ yield idempotents $e(X \leftarrow U \rightarrow Y) \in \text{End } \phi(X, Y)$, which we now denote by $e^L(X \leftarrow U \rightarrow Y)$. Similarly the right-sided operation $.Z$ yield idempotents $e^R(X \leftarrow V \rightarrow Y)$.

Proposition 10.1. $e^L(X \leftarrow U \rightarrow Y)$ and $e^R(X \leftarrow V \rightarrow Y)$ commute with each other.

Proof. $e^L(X \xleftarrow{a} U \xrightarrow{b} Y) = \{X \xleftarrow{1} X \xrightarrow{1} X, X \xleftarrow{a} U \xrightarrow{b} Y\}$ is the composite

$$\phi(X, Y) \xrightarrow{X.} \phi(XX, XY) \xrightarrow{\langle \Delta, (a, b) \rangle^*} \phi(X, U) \xrightarrow{\langle 1, b \rangle^*} \phi(X, Y)$$

and $e^R(X \xleftarrow{c} V \xrightarrow{d} Y) = \{Y \xleftarrow{d} V \xrightarrow{c} X, Y \xleftarrow{1} Y \xrightarrow{1} Y\}$ is the composite

$$\phi(X, Y) \xrightarrow{.Y} \phi(XY, YY) \xrightarrow{\langle (c, d), \Delta \rangle^*} \phi(V, Y) \xrightarrow{\langle c, 1 \rangle^*} \phi(X, Y).$$

So $e^R(X \leftarrow V \rightarrow Y) \circ e^L(X \leftarrow U \rightarrow Y)$ is the composite of the upper-right path of the commutative diagram

$$\begin{array}{ccccccc} \phi(X, Y) & \xrightarrow{X.} & \phi(XX, XY) & \xrightarrow{\langle \Delta, (a, b) \rangle^*} & \phi(X, V) & \xrightarrow{\langle 1, b \rangle^*} & \phi(X, Y) \\ & & .Y \downarrow & & .Y \downarrow & & .Y \downarrow \\ & & \phi(XXY, XYY) & \xrightarrow{\langle \Delta Y, (a, b) Y \rangle^*} & \phi(XY, VY) & \xrightarrow{\langle 1, bY \rangle^*} & \phi(XY, YY) \\ & & & & & & \downarrow \langle (c, d), \Delta \rangle^* \\ & & & & & & \phi(V, Y) \\ & & & & & & \downarrow \langle 1, c \rangle^* \\ & & & & & & \phi(X, Y) \end{array}$$

Now we have commutative diagrams

$$\begin{array}{ccc} XXY & \xleftarrow{\Delta Y} & XY \\ (c, c, d) \nearrow & & \uparrow (c, d) \\ & & V \end{array}$$

$$\begin{array}{ccccc}
XY Y & \xleftarrow{(a,b)Y} & UY & \xrightarrow{bY} & YY \\
& (a,b,b) \nearrow & \uparrow (a,b) & & \uparrow \Delta \\
& & U & \xrightarrow[b]{} & Y
\end{array}$$

with the square pull-back. Hence $e^R(X \leftarrow V \rightarrow Y) \circ e^L(X \leftarrow U \rightarrow Y)$ is equal to the composite

$$\begin{array}{c}
\phi(X, Y) \xrightarrow{X.} \phi(XX, XY) \xrightarrow{.Y} \phi(XXY, XYY) \\
\langle (c, c, d), (a, b, b) \rangle^* \downarrow \\
\phi(V, U) \\
\langle c, b \rangle_* \downarrow \\
\phi(X, Y)
\end{array}$$

On the other hand, $e^L(X \leftarrow U \rightarrow Y) \circ e^R(X \leftarrow V \rightarrow Y)$ is the composite of the upper-right path of the commutative diagram

$$\begin{array}{ccccccc}
\phi(X, Y) & \xrightarrow{.Y} & \phi(XY, YY) & \xrightarrow{\langle (c, d), \Delta \rangle^*} & \phi(V, Y) & \xrightarrow{\langle c, 1 \rangle^*} & \phi(X, Y) \\
& & X. \downarrow & & X. \downarrow & & X. \downarrow \\
\phi(XXY, XYY) & & \xrightarrow{\langle X(c, d), X\Delta \rangle^*} & \phi(XV, XY) & \xrightarrow{\langle Xc, 1 \rangle^*} & \phi(XX, XY) & \\
& & & & & & \langle \Delta, (a, b) \rangle^* \downarrow \\
& & & & & & \phi(X, V) \\
& & & & & & \langle b, 1 \rangle_* \downarrow \\
& & & & & & \phi(X, Y)
\end{array}$$

We have commutative diagrams

$$\begin{array}{ccc}
XY Y & \xleftarrow{X\Delta} & XY \\
& (a,b,b) \nearrow & \uparrow (a,b) \\
& & U
\end{array}$$

$$\begin{array}{ccccc}
XXY & \xleftarrow{X(c,d)} & XV & \xrightarrow{Xc} & XX \\
& (c,c,d) \nearrow & \uparrow (c,1) & & \uparrow \Delta \\
& & V & \xrightarrow{c} & X
\end{array}$$

Hence $e^L(X \leftarrow U \rightarrow Y) \circ e^R(X \leftarrow V \rightarrow Y)$ is equal to the composite

$$\begin{aligned}
\phi(X, Y) &\xrightarrow{.Y} \phi(XX, XY) \xrightarrow{X.} \phi(XXY, XYY) \\
&\quad \langle (c, c, d), (a, b, b) \rangle^* \downarrow \\
&\quad \phi(V, U) \\
&\quad \langle c, b \rangle_* \downarrow \\
&\quad \phi(X, Y)
\end{aligned}$$

By the commutativity of $X.$ and $.Y$ it follows that

$$e^R(X \leftarrow V \rightarrow Y) \circ e^L(X \leftarrow U \rightarrow Y) = e^L(X \leftarrow U \rightarrow Y) \circ e^R(X \leftarrow V \rightarrow Y).$$

11. equivalence ${}_S\mathbb{M}(\mathcal{S}, \mathcal{S})_S \simeq \mathbb{N}(\mathcal{H})$

We will construct an equivalence ${}_S\mathbb{M}(\mathcal{S}, \mathcal{S})_S \simeq \mathbb{N}(\mathcal{H})$. Let ϕ be an object of ${}_S\mathbb{M}(\mathcal{S}, \mathcal{S})_S$. For an object

$$\mathbf{X} = \left(\begin{array}{ccc} & U & \\ \swarrow & & \searrow \\ X & & Y \\ \nwarrow & & \nearrow \\ & V & \end{array} \right)$$

in \mathcal{H} , let

$$e(\mathbf{X}) = e^L(X \leftarrow U \rightarrow Y) \circ e^R(X \leftarrow V \rightarrow Y)$$

By Proposition 10.1, $e(\mathbf{X})$ is an idempotent endomorphism on $\phi(X, Y)$. Define

$$\theta(\mathbf{X}) = \text{Im } e(\mathbf{X}).$$

For a morphism $\mathbf{f}: \mathbf{X} \rightarrow \mathbf{X}'$ in \mathcal{H} , define the maps

$$\mathbf{f}_*: \theta(\mathbf{X}) \rightarrow \theta(\mathbf{X}')$$

$$\mathbf{f}^*: \theta(\mathbf{X}') \rightarrow \theta(\mathbf{X})$$

in a similar way to Section 8.

We can verify θ is an object of $\mathbb{N}(\mathcal{H})$.

Theorem 11.1. *The functor $\phi \mapsto \theta$ gives an equivalence ${}_S\mathbb{M}(\mathcal{S}, \mathcal{S})_S \rightarrow \mathbb{N}(\mathcal{H})$*

12. category \mathcal{H}_0 and equivalence $\mathbb{N}(\mathcal{H}) \simeq \mathbb{M}(\mathcal{H}_0)$

A G -set is said to be connected if it consists of a single orbit. Let \mathcal{H}_0 be the full subcategory of \mathcal{H} consisting of objects which are direct sums of objects

$$\begin{array}{ccc}
& U & \\
\swarrow & & \searrow \\
X & & Y \\
\nwarrow & & \nearrow \\
& V &
\end{array}$$

such that all arrows are isomorphisms and X is connected.

Theorem 12.1. *The inclusion functor $\mathcal{H}_0 \rightarrow \mathcal{H}$ has a right adjoint R .*

Proof is omitted. Let $\rho_X: R(X) \rightarrow X$ be the canonical morphism of adjoint.

Corollary 12.2. *\mathcal{H}_0 has finite projective limits.*

Define a category \mathcal{T} as follows. An object is a pair (X, a) of a G -set X and an automorphism $a: X \rightarrow X$. Morphisms are defined naturally. Let \mathcal{T}_0 be a full subcategory of \mathcal{T} consisting of objects which are direct sums of objects (X, a) such that X is connected.

Proposition 12.3. *We have an equivalence $\mathcal{H}_0 \simeq \mathcal{L}_0$*

Proposition 12.4. *If $\phi \in \mathbb{N}(\mathcal{H})$, then*

$$\rho_{X*}: \phi(R(X)) \rightarrow \phi(X)$$

are isomorphisms for all $X \in \mathcal{H}$.

Since \mathcal{H}_0 has pull-backs, we can speak about Mackey functors on \mathcal{H}_0 . Let $\mathbb{M}(\mathcal{H}_0)$ denote the category of Mackey functors on \mathcal{H}_0 .

Theorem 12.5. *We have an equivalence*

$$\mathbb{N}(\mathcal{H}) \simeq \mathbb{M}(\mathcal{H}_0).$$

Combining this with the equivalences

$${}_{\mathcal{M}}\mathbb{B}(\mathcal{M}, \mathcal{M})_{\mathcal{M}} \simeq {}_{\mathcal{S}}\mathbb{M}(\mathcal{S}, \mathcal{S})_{\mathcal{S}} \simeq \mathbb{N}(\mathcal{H})$$

and

$$\mathbb{M}(\mathcal{H}_0) \simeq \mathbb{M}(\mathcal{T}_0),$$

we obtain

Theorem 12.6.

$${}_{\mathcal{M}}\mathbb{B}(\mathcal{M}, \mathcal{M})_{\mathcal{M}} \simeq \mathbb{M}(\mathcal{T}_0).$$

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