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Preface

The theory of quadratic forms is a basic concept in mathematics, also it appears in various branches of sciences. One of its extensions is the theory of *numerical range*. Let $\mathbb{C}^n$ be the n-tuple column vectors with complex entries. For any $n \times n$ complex matrix A, the numerical range of A is defined by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}.$$

The numerical range was named by Marshall Stone in 1932. Originally, Toeplitz (1918) and Hausdorff (1919) called it the *Wertvorrat* of a bilinear form [23]. This set and its generalizations have been studied extensively in the past one century. The convexity is the most important property of the numerical range. In addition, these subjects are related and have applications to many different areas of nature science such as functional analysis, numerical analysis, systems theory, quantum physics, nuclear magnetic resonance (NMR) [21] etc. In this thesis, we will concern some related problems of the classical numerical range and its generalizations.

As mentioned in the previous paragraph, generalized numerical ranges have been extensively studied recently. One of these generalizations is the *k-numerical range* $W_k(A)$ for a given $n \times n$ complex matrix A. This set is defined as

$$W_k(A) = \left\{ \sum_{i=1}^{k} (UAU^*)_{ii} : U \text{ is unitary} \right\}.$$

The k-numerical ranges were first studied by Paul Halmos. The convexity of this set was also conjectured by Halmos and proved by his Ph.D student Berger later. From its definition, it turns out that whether this set is convex when we replace the summation with the multiplication. However, this range is not convex even for the three-by-three case. This subject was first investigated by Marvin Marcus in 1972. Moreover the original definition is more general and abstract. In Chapter 1, we consider this generalized numerical range, namely product range. This range is defined as

$$W_k^{\Pi}(A) = \left\{ \prod_{i=1}^{k} (UAU^*)_{ii} : U \text{ is unitary} \right\}, \ 1 \le k \le n,$$

where A is an $n \times n$ complex matrix. When $k = 2$ and A is a 3×3 normal matrix, we prove the product range is convex if the eigenvalues of A form an acute-angled triangle inscribed to the unit circle.

It is well known that a real symmetric matrix can be diagonalized by an orthogonal matrix. However, this result is not true for a general complex symmetric matrix. For this reason, the factorization for a complex matrix has been investigated by many mathematicians including Schur (1945), Hua (1944), Siegel (1943), Jacobsen (1939) and Takagi (1925). Historical priority should apparently to be Léon Autonne in 1915. In addition, Takagi provided a factorization for a complex symmetric matrix [16, 26]. Recently, a number of other authors have studied on complex symmetric operators or matrices. Moreover, this subject appears frequently in the study of damped vibrations of linear systems [12] and physics [17]. In Chapter 2, we treat the relations between complex symmetric matrices and classical numerical ranges. We characterize matrices whose discrete Fourier transforms are complex symmetric. The Fourier transform defines a unitary similarity. The classical numerical range $W(A)$ of a matrix is invariant under unitary similarities, that is, $W(U^*AU) = W(A)$ for all unitary matrices U. So we found a family of matrices which have the same classical numerical range.

The *q-numerical range* of given $A \in M_n$ and $q \in \mathbb{C}$ with $|q| \le 1$ is defined as the set

$$W_q(A) = \{\eta^* A\xi : \xi, \eta \in \mathbb{C}^n, \xi^*\xi = \eta^*\eta = 1, \eta^*\xi = q\}.$$

If $q = 1$, the range $W_1(A)$ is reduced to $W(A)$. Boundary points of the numerical range $W(A)$ of an $n \times n$ matrix A lie on an algebraic curve of degree less than $n(n-1)$ or its bitangents. Also the boundary points of $W_q(A)$ also lie on an algebraic curve. However, its degree is supposed to be so high. Indeed, the degree of the boundary of $W_q(A)$ is not greater than $N(N-1)^2$ where $N = 2n(n-1)^2$. Accordingly, it is hard to compute the boundary equation of the q-numerical range for a given matrix A by using a standard personal computer even for the three-by-three case. In Chapter 3, the final chapter, we shall consider the boundary points of the q-numerical range for a special Toeplitz nilpotent matrix . We also provide a program to plot $W_q(A)$ by using “Mathematica”.

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Notation

$\mathbb{C}^n$	the n-tuple column vectors with complex entries
M_n	$n \times n$ matrices with complex entries
$M_n(\mathbb{R})$	$n \times n$ matrices with real entries
O_n	$n \times n$ zero matrix
I_n	$n \times n$ identity matrix
$J_n(\lambda)$	$n \times n$ elementary Jordan block with diagonal entries λ
A^{-1}	inverse of matrix A
A^T	transpose of matrix A
$\mathrm{rank}(A)$	rank of matrix A
$\mathrm{tr}(A)$	trace of matrix A
$\det(A)$	determinant of matrix A
$\nu(A)$	index of matrix A
$N(A)$	null space of matrix A
$\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$	diagonal matrix with $\lambda_1, \lambda_2, \ldots, \lambda_n$ on the diagonal
$\mathrm{U}(n)$	$n \times n$ unitary matrix
$\mathbf{Re}(z)$	the real part of z
$\mathbf{Im}(z)$	the imaginary part of z
$\arg(z)$	the argument of z

Chapter 1

The 2-Product Range for Normal Matrices of Order Three

1.1 Introduction

In these three decades the study of classical and generalized numerical ranges of matrices has produced various interesting results. The shape of numerical ranges is discussed by many authors (cf. [10, 11, 19, 20, 31, 35, 42, 44, 47]). An entrywise nonnegative $n \times n$ matrix (a_{ij}) is called *doubly stochastic* if all its row and column sums are 1. Majorization theory related with doubly stochastic matrices plays an important role in the theory of inequalities. A special class of doubly stochastic matrices called unistochastic type are used in mathematical physics (cf. [4, 13, 14]). An $n \times n$ real matrix $A = (a_{ij})$ is said to be *unistochastic* if it is expressed as

$$a_{ij} = |u_{ij}|^2,$$

for some $n \times n$ unitary matrix $U = (u_{ij})$, where U can be chosen so that $\det(U) = 1$. A unistochastic matrix is also called *orthostochastic* in some branches of mathematics. Throughout this Chapter, we denote by $\mathrm{Uni}(n)$, the set of unistochastic matrices. However the structure of $\mathrm{Uni}(n)$ is related to the Lie group $\mathrm{U}(n)$, an efficient structure theory of $\mathrm{Uni}(n)$ for $n \geq 4$ has not been found yet. Au-Yeung and Poon [1] provided the structure theorem of $\mathrm{Uni}(3)$ by using inequalities on the entries a_{ij} of the matrix $(a_{ij}) \in \mathrm{Uni}(3)$. We denote by $\mathrm{D}(3)$ the semi-group of all 3×3 doubly stochastic matrices. The following two characterizations are very useful to recognize the geometric structure of the set $\mathrm{Uni}(3)$.

Theorem 1.1. [Au-Yeung, Poon [1]] *A* 3×3 *doubly stochastic matrix* (a_{ij}) *belongs to* Uni(3) *if and only if the* 3 *quantities* $\sqrt{a_{11}a_{21}}$, $\sqrt{a_{12}a_{22}}$, $\sqrt{a_{13}a_{23}}$ *satisfy the inequalities*

$$\begin{cases} \sqrt{a_{11}a_{21}} \le \sqrt{a_{12}a_{22}} + \sqrt{a_{13}a_{23}}, \\ \sqrt{a_{12}a_{22}} \le \sqrt{a_{11}a_{21}} + \sqrt{a_{13}a_{23}}, \\ \sqrt{a_{13}a_{23}} \le \sqrt{a_{11}a_{21}} + \sqrt{a_{12}a_{22}}. \end{cases}$$

Theorem 1.2. [Au-Yeung, Poon [1]] *The set* Uni(3) *is star-shaped with respect to the Van der Waerden matrix*

$$C_0 = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

Let $A(3)$ be the real affine space of the 3×3 real matrices with all row and column sums equal to 1. Denote by ∂Uni(3), the boundary of Uni(3) in $A(3)$ with the usual metric topology. A group theoretic characterization of Uni(3) is given by

$$\partial\mathrm{Uni}(3) = \{(g_{ij}^2) : (g_{ij}) \in SO(3)\},$$

$$\mathrm{Uni}(3) = \{tC_0 + (1-t)A : A \in \partial\mathrm{Uni}(3), 0 \le t \le 1\},$$

where $SO(3)$ is the group of rotations in the 3-dimensional Euclidean space [36, 38]. By the relations,

$$\begin{cases} a_{13} = 1 - a_{11} - a_{12}, \\ a_{23} = 1 - a_{21} - a_{22}, \\ a_{31} = 1 - a_{11} - a_{21}, \\ a_{32} = 1 - a_{12} - a_{22}, \\ a_{33} = a_{11} + a_{12} + a_{21} + a_{22} - 1, \end{cases}$$

we can treat a_{11}, a_{12}, a_{21} and a_{22} as independent variables of $A(3)$. Under this convention, a matrix $(a_{ij}) \in D(3) \subset A(3)$ belongs to ∂Uni(3) if and only if

$$(1 - a_{11} - a_{12} - a_{21} - a_{22} + a_{11}a_{22} + a_{12}a_{21})^2 - 4a_{11}a_{12}a_{21}a_{22} = 0,$$

(cf. [39]).

In addition, using the inequality in Theorem 1.1, Au-Yeung and Poon gave the necessary and sufficient condition of the convexity of the range $W_C(A)$ for 3×3 normal matrices A, C [1] where

$$W_C(A) = \{\mathrm{tr}(CUAU^*) : U \in \mathrm{U}(3)\}.$$

This range is one-type of generalization of the classical numerical range $W(A)$ of an $n \times n$ matrix defined as

$$W(A) = \{\xi^* A \xi : \xi \in \mathbb{C}^n, \xi^*\xi = 1\}.$$

We refer the reader to [15] for fundamental properties of the numerical ranges.

We shall consider another subject on Uni(3) which also concerns with the geometric property of this set. In this chapter, we will consider the shape of the range

$$W_k^{\Pi}(N) = \left\{ \prod_{i=1}^{k} (UNU^*)_{ii} : U \in \mathrm{U}(n) \right\}, \ 1 \le k \le n, \tag{1.1.1}$$

when $k = 2$ and N is a 3×3 normal matrix. The similarity relation

$$W_2^{\Pi}(\mathrm{diag}(\alpha\lambda_1, \alpha\lambda_2, \alpha\lambda_3)) = \alpha^2 W_2^{\Pi}(\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)) \tag{1.1.2}$$

is useful to treat this range.

In the case $k = 1$, $W_1^{\Pi}(N)$ is the convex hull of $\lambda_1, \lambda_2, \ldots, \lambda_n$. In the case $n = 2$, every doubly stochastic matrix is unistochastic. For $N = \mathrm{diag}(\lambda_1, \lambda_2)$, we get the line segment

$$W_2^{\Pi}(N) = \left[\lambda_1\lambda_2, \left(\frac{\lambda_1 + \lambda_2}{2} \right)^2 \right].$$

The investigation of $W_k^{\Pi}(N)$, $k = 1, \cdots, n$, was firstly proposed by Marcus [34]. The convexity of the range $W_k^{\Pi}(N)$ is characterized in [5] in the case the eigenvalues of N are on a straight line. Various examples of non-convex $W_k^{\Pi}(N)$ are given in [38].

The main result of this chapter is Theorem 1.4 which provides a wide class of 3×3 diagonal matrices $N = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ for which

$$W_2^{\Pi}(N) = \mathrm{Conv}(\lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3),$$

where $\mathrm{Conv}(\lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3)$ is the convex hull of the three points $\lambda_1\lambda_2$, $\lambda_2\lambda_3$ and $\lambda_1\lambda_3$ (cf. Corollary 1.5 and 1.6). The standard methods of convex analysis and the usage of the positive semi-definiteness of functionals are the main tools of this chapter to study the subject.

1.2 Preliminaries

By the definition of $W_k^\Pi(N)$, when N is normal we may assume

$$N = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$$

in (1.1.1) where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of N. Moreover, having in mind that

$$\prod_{i=1}^{k}(UNU^*)_{ii} = \prod_{i=1}^{k}\sum_{j=1}^{n}|u_{ij}|^2\lambda_j,$$

we can define $W_k^\Pi(N)$ equivalently by

$$W_k^\Pi(N) = \left\{\prod_{i=1}^{k}\sum_{j=1}^{n}a_{ij}\lambda_j : (a_{ij}) \in \mathrm{Uni}(n)\right\}. \tag{1.2.1}$$

In the case N is a 3×3 normal matrix and $k = 2$, the 3 eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are on a straight line ℓ, we can characterize the range $W_2^\Pi(N)$ according to the different two situations: (i) $0 \in \ell$, (ii) $0 \notin \ell$.

In the case (i), we may assume that $\lambda_1, \lambda_2, \lambda_3$ are real numbers and (i-1) $0 \le \lambda_3 \le \lambda_2 \le \lambda_1$ or (i-2) $\lambda_3 \le 0 \le \lambda_2 \le \lambda_1$. The equations

$$W_2^\Pi(N) = [\lambda_2\lambda_3, (\lambda_1+\lambda_2)^2/4], \tag{1.2.2}$$

and

$$W_2^\Pi(N) = [\lambda_1\lambda_3, (\lambda_1+\lambda_2)^2/4], \tag{1.2.3}$$

hold respectively in the case (i-1) and (i-2).

In the case (ii), the range $W_2^\Pi(N)$ is not convex by [5, Theorem 3.1]. The exact figure of $W_2^\Pi(N)$ in this situation is given in [38] under the assumption $\mathbf{Im}(\lambda_1) = \mathbf{Im}(\lambda_2) = \mathbf{Im}(\lambda_3) = 1$. The range $W_2^\Pi(N)$ is characterized as the image of $D(3)$ under a quadratic map. So we may assume that $(\lambda_2 - \lambda_3)/(\lambda_1 - \lambda_3)$ is a well defined imaginary number. We assume that $\lambda_1, \lambda_2, \lambda_3$ lie on a circle counterclockwisely. The relations (1.2.2), (1.2.3) and some numerical experiments suggest the equation

$$W_2^\Pi(\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)) = \mathrm{Conv}(\lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3), \tag{1.2.4}$$

holds only when the the interior Γ of the circumscribed circle of $\triangle\lambda_1\lambda_2\lambda_3$ satisfies

$$0 \in \Gamma \tag{1.2.5}$$

and

$$0 < \arg\left(\frac{\lambda_3-\lambda_1}{\lambda_2-\lambda_1}\right), \arg\left(\frac{\lambda_1-\lambda_2}{\lambda_3-\lambda_2}\right), \arg\left(\frac{\lambda_2-\lambda_3}{\lambda_1-\lambda_3}\right) \le \frac{\pi}{2}. \tag{1.2.6}$$

However, so far we are not sure if these two conditions are necessary for the equation (1.2.4) to hold, we show its necessity of the condition (1.2.6) under some special two situations in Section 5. We assume these conditions to consider the subject. The next lemma provides the main motivation to assume the condition (1.2.5). For a moment, we assume that the three points $\lambda_1\lambda_2$, $\lambda_1\lambda_3$, $\lambda_2\lambda_3$ are not colinear. For any $1 \leq i < j \leq 3$, let $k = \{1, 2, 3\}\backslash\{i, j\}$. Denote by H_{ij}, the closed half-plane with the boundary passing through $\lambda_i\lambda_k$, $\lambda_j\lambda_k$ satisfying $\lambda_i\lambda_j \in H_{ij}$. We have the following lemma.

Lemma 1.3. *Let $\lambda_1, \lambda_2, \lambda_3$ be mutually distinct non-zero complex numbers satisfying the condition* (1.2.5)*. Then the three points $\lambda_1\lambda_2$, $\lambda_1\lambda_3$, $\lambda_2\lambda_3$ are not colinear. Moreover the interior of the respective three half-planes H_{12}, H_{13}, H_{23} contains the respective points $\lambda_3^2, \lambda_2^2, \lambda_1^2$.*

Proof. We shall prove the first part of the lemma. We set $\lambda_0 = \lambda_1\lambda_2\lambda_3 \neq 0$. Then we have the relation

$$\lambda_1\lambda_2 = \frac{\lambda_0}{\lambda_3}, \quad \lambda_1\lambda_3 = \frac{\lambda_0}{\lambda_2}, \quad \lambda_2\lambda_3 = \frac{\lambda_0}{\lambda_1}.$$

The three points $\lambda_1\lambda_2, \lambda_1\lambda_3, \lambda_2\lambda_3$ are on a line ℓ with $0 \notin \ell$ if and only if their inverses

$$\frac{\lambda_3}{\lambda_0}, \quad \frac{\lambda_2}{\lambda_0}, \quad \frac{\lambda_1}{\lambda_0}$$

lie on a common circle passing through 0, but this is impossible by the assumption (1.2.5). We recognize this assertion by the fact that the condition

$$\lambda_1 = 1 + e^{i\theta_1}, \quad \lambda_2 = 1 + e^{i\theta_2}, \quad \lambda_3 = 1 + e^{i\theta_3}$$

implies

$$\frac{\lambda_2\lambda_3 - \lambda_1\lambda_2}{\lambda_1\lambda_3 - \lambda_1\lambda_2} = \frac{\sin((\theta_3 - \theta_1)/2)}{\sin((\theta_3 - \theta_2)/2)} \frac{\cos(\theta_2/2)}{\cos(\theta_1/2)}.$$

We shall prove the second part of the lemma. By exchange the roles of $\lambda_1, \lambda_2, \lambda_3$, it is sufficient to prove that the point λ_3^2 belongs to the interior of H_{12}. To prove this assertion, we use the following setting:

$$\lambda_j = e^{i\theta}(\mu_j - \mu_0) \tag{1.2.7}$$

for $j = 1, 2, 3$ and $\theta \in \mathbb{R}$ where $\mu_1 = iy_1$, $\mu_2 = iy_2$, $\mu_3 = a$ $(y_1 \geq y_2, a > 0)$ and $\mu_0 = \alpha + i\beta$. Then a point $z \in \mathbb{C}$ belongs to the open disc bounded by the circle circumscribed to $\triangle\mu_1\mu_2\mu_3$ if and only if

$$-a\mathbf{Re}(z)^2 - a\mathbf{Im}(z)^2 + (a^2 + y_1y_2)\mathbf{Re}(z) + (ay_1 + ay_2)\mathbf{Im}(z) - ay_1y_2 > 0,$$

in other words, the assumption on $\mu_0 = \alpha + i\beta$ is written as

$$C(\alpha, \beta) = -a\alpha^2 - a\beta^2 + a^2\alpha + y_1y_2\alpha + ay_1\beta + ay_2\beta - ay_1y_2 > 0.$$

By using the point $\mu_0 = \alpha + i\beta$, the condition $0 \in \Gamma$ is expressed as $C(\alpha, \beta) > 0$. To prove the fact $\lambda_3^2 \in \text{Interior}(H_{12})$, we consider the relative positions of the four points

$$e^{-i\theta}\lambda_1, e^{-i\theta}\lambda_2, e^{-i\theta}\lambda_3, \frac{e^{-i\theta}\lambda_1\lambda_2}{\lambda_3}.$$

It is sufficient to prove that

$$\mathbf{Re}(e^{-i\theta}(\lambda_3 - \lambda_1))\mathbf{Re}(e^{-i\theta}(\frac{\lambda_1\lambda_2}{\lambda_3} - \lambda_1)) > 0.$$

By direct computations, we have

$$((\alpha-a)^2+\beta^2)\,\mathbf{Re}(e^{-i\theta}(\frac{\lambda_1\lambda_2}{\lambda_3}-\lambda_1)) = -a\alpha^2-a\beta^2+a^2\alpha+y_1y_2\alpha+ay_1\beta+ay_2\beta-ay_1y_2 > 0,$$

$$\mathbf{Re}(e^{-i\theta}(\lambda_3 - \lambda_1)) = a > 0,$$

where $(\alpha - a)^2 + \beta^2 > 0$ by the assumption on μ_0. Thus the second part of the lemma is proved. □

Under the setting (1.2.7), let $\tilde{\theta} = \arg(\lambda_3)$. Then the argument $\xi = \theta + \tilde{\theta}$ satisfies

$$\arg(e^{-i\xi}(\lambda_1\lambda_3 - \lambda_2\lambda_3)) \equiv \frac{\pi}{2} \mod 2\pi.$$

Especially, if $\theta = 0$ then $\xi = \arg(\lambda_3)$ satisfies this relation.

1.3 Main results

We shall formulate the main theorem of this chapter. Its contents are related with an acute-angled or a right-angled triangle $\triangle\mu_1\mu_2\mu_3$ and its circumcenter ζ. We consider the circular disc $\tilde{\Gamma}_i$ with center ζ for which the line segment $[\mu_j, \mu_k]$ $(i, j, k \in \{1, 2, 3\},\ i \neq j,\ i \neq k$ and $j \neq k)$ is a tangent of $\tilde{\Gamma}_i$. In the case $\triangle\mu_1\mu_2\mu_3$ is a right-angled triangle one of the circular discs $\tilde{\Gamma}_j$ is reduced to a point. To express the role of the angles $\angle\mu_j\mu_i\mu_k$ we represent the positions of μ_i by using these angles. The main theorem is the following.

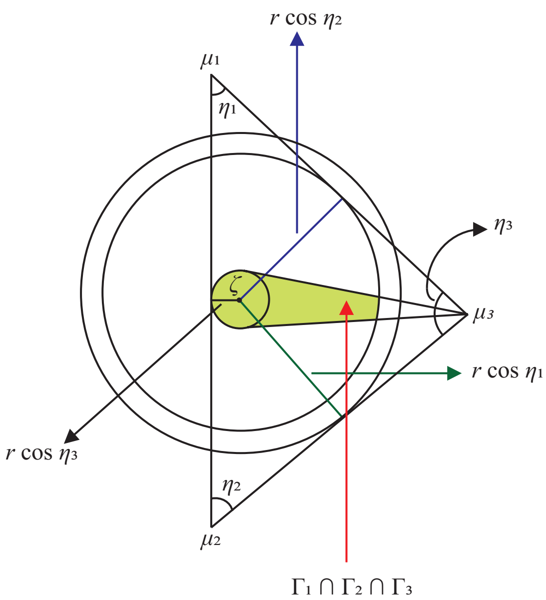

Figure 1. ζ is the circumcenter of $\triangle\mu_1\mu_2\mu_3$.

Theorem 1.4. *Suppose that μ_1, μ_2 and μ_3 are three complex numbers given by*

$$\begin{aligned} \mu_2 &= \zeta + re^{i\rho} = \zeta + re^{i(\rho+2\eta_1+2\eta_2+2\eta_3)}, \\ \mu_3 &= \zeta + re^{i(\rho+2\eta_1)}, \\ \mu_1 &= \zeta + re^{i(\rho+2\eta_1+2\eta_2)}, \end{aligned}$$

which satisfy $0 < \eta_1, \eta_2, \eta_3 \le \pi/2$, $\eta_1 + \eta_2 + \eta_3 = \pi$, $r > 0$, $0 \le \rho \le 2\pi$ and $\zeta \in \mathbb{C}$. Let

$$\tilde{\Gamma}_j = \{z \in \mathbb{C} : |z - \zeta| \le r\cos\eta_j\}$$

and $\Gamma_j = \mathrm{Conv}(\tilde{\Gamma}_j, \mu_j)$ for $j = 1, 2, 3$. Then the triple

$$(\lambda_1, \lambda_2, \lambda_3) = (\mu_1 - \mu_0, \mu_2 - \mu_0, \mu_3 - \mu_0)$$

for a point $\mu_0 = \alpha + i\beta$ with $|\mu_0 - \zeta| < r$ satisfies

$$W_2^{\Pi}(\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)) = \mathrm{Conv}(\lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3)$$

if and only if $\mu_0 \in \Gamma_1 \cap \Gamma_2 \cap \Gamma_3$. The region $\Gamma_1 \cap \Gamma_2 \cap \Gamma_3$ is shown in Figure 1.

The relative position of the four points 0, λ_1, λ_2 and λ_3 is crucial to determine the range $W_2^{\Pi}(\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3))$. If we apply Theorem 1.4 to the normal matrices $N_1 = \mathrm{diag}(1, i, -i)$ and $N_2 = N_1 - \frac{1}{2}I_3$, then

$$W_2^{\Pi}(N_1) = \mathrm{Conv}(1, i, -i)$$

and

$$W_2^{\Pi}(N_2) = \mathrm{Conv}\left(\frac{5}{4}, -\frac{1}{4}+\frac{i}{2}, -\frac{1}{4}-\frac{i}{2}\right)$$

are not similar.

As simple consequences of Theorem 1.4 we provide the following two corollaries.

Corollary 1.5. *Suppose that $\lambda_1, \lambda_2, \lambda_3$ are complex numbers with modulus* 1 *and satisfies the condition* (1.2.6). *Then an acute-angled or a right-angled triangle $\triangle\lambda_1\lambda_2\lambda_3$ satisfies*

$$W_2^{\Pi}(\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)) = \mathrm{Conv}(\lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3).$$

Proof. The assumption of this corollary corresponds to the case μ_0 is the circumcenter of $\triangle\mu_1\mu_2\mu_3$ in Theorem 1.4. This point belongs to the circular disc $\tilde{\Gamma}_j$ for $j = 1, 2, 3$. □

Corollary 1.6. *Suppose that $\lambda_1, \lambda_2, \lambda_3$ are vertices of an acute-angled or a right-angled triangle in the Gaussian plane and satisfy the condition $\lambda_1+\lambda_2+\lambda_3 = 0$. Then the relation*

$$W_2^{\Pi}(\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)) = \mathrm{Conv}(\lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3)$$

holds.

The assumption of this corollary corresponds to the case $\mu_0 = (\mu_1 + \mu_2 + \mu_3)/3$, that is μ_0 is the centroid of $\triangle\mu_1\mu_2\mu_3$ in Theorem 1.4. We shall prove μ_0 belongs to $\Gamma_1 \cap \Gamma_2 \cap \Gamma_3$ after the proof of Theorem 1.4 in Section 4.

To prove Theorem 1.4, we can assume that the line passing through μ_1 and μ_2 is parallel to the imaginary axis by a suitable choice of the angle ρ by virtue of (1.1.2). We may also assume that

$$\mathbf{Re}(\mu_1) = \mathbf{Re}(\mu_2) < \mathbf{Re}(\mu_3).$$

The set $\partial\tilde{\Gamma}_1$ can be viewed as the circle with the center at the circumcenter of $\triangle\mu_1\mu_2\mu_3$ and the line segment $[\mu_2, \mu_3]$ is a tangent of $\partial\tilde{\Gamma}_1$. The range $W_2^{\Pi}(N)$ does not satisfy a simple property under the replacement of N by $N + \alpha I$ ($\alpha \in \mathbb{C}$). We shall see later (the argument after (1.4.7)), we may assume that

$$0 = \mathbf{Re}(\mu_1) = \mathbf{Re}(\mu_2) < \mathbf{Re}(\mu_3) \text{ and } \mathbf{Im}(\mu_3) = 0.$$

Since $\triangle\mu_1\mu_2\mu_3$ is acute-angled or right-angled, we may assume that

$$\mathbf{Im}(\mu_1) \geq 0 \text{ and } \mathbf{Im}(\mu_2) \leq 0.$$

The replacement of $\{\mu_1, \mu_2, \mu_3\}$ by $\{\mu_1 + \alpha, \mu_2 + \alpha, \mu_3 + \alpha\}$ means the replacement of

$$\tilde{\Gamma_j} = \{z \in \mathbb{C} : |z - \zeta| \leq r \cos \eta_j\}$$

by

$$\tilde{\Gamma_j} = \{z \in \mathbb{C} : |z - (\zeta + \alpha)| \leq r \cos \eta_j\}.$$

By this reason, we will use the translation of μ_j for simple expression of the coefficients of rational functions of $\mathbf{Re}(\mu_j)$ and $\mathbf{Im}(\mu_j)$.

Remark 1.7. *One would suppose that the assertion of Corollary 1.5 remains valid if we replace the circumcenter by the incenter of a triangle. However such an assertion is false. The equation (1.2.4) does not necessary hold under the assumption*

$$0 < \arg((\lambda_3-\lambda_1)/(\lambda_2-\lambda_1)), \arg((\lambda_1-\lambda_2)/(\lambda_3-\lambda_2)), \arg((\lambda_2-\lambda_3)/(\lambda_1-\lambda_3)) \leq \pi/2,$$

and the origin is the incenter of $\triangle\lambda_1\lambda_2\lambda_3$*. For example, when*

$$N = \mathrm{diag}(\frac{75i}{4}, 4 - \frac{21i}{4}, -4 - \frac{21i}{4}),$$

(1.2.4) does not hold.

1.4 Proofs of results

For the proof of Theorem 1.4, we firstly prove the following inclusion

$$W_2^{\Pi}(N) \subseteq \mathrm{Conv}(\lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3), \tag{1.4.1}$$

where $N = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$.

By using the half-planes H_{ij} introduced in Lemma 1.3, the inclusion (1.4.1) is rewritten as

$$W_2^{\Pi}(N) \subseteq H_{12} \cap H_{23} \cap H_{13}.$$

We shall prove the inclusion

$$W_2^{\Pi}(N) \subseteq H_{12}, \tag{1.4.2}$$

under the assumption $\mu_0 \in \Gamma_3$. Similar inclusions for H_{23} and H_{13} can be proved in the same manner under the assumption $\mu_0 \in \Gamma_1$ and $\mu_0 \in \Gamma_2$, respectively. To prove the inclusion (1.4.2), we use a functional on the set Uni(3). Let $\xi = \arg(\lambda_1\lambda_3 - \lambda_2\lambda_3) - \pi/2$. Then this angle satisfies

$$\mathbf{Re}(e^{-i\xi}\lambda_1\lambda_3) = \mathbf{Re}(e^{-i\xi}\lambda_2\lambda_3) < \mathbf{Re}(e^{-i\xi}\lambda_1\lambda_2).$$

For any $(a_{ij}) \in \mathrm{Uni}(3)$, the element z of $W_2^{\Pi}(N)$ can be expressed as

$$\begin{aligned} z &= \lambda_1\lambda_2(a_{11}a_{22} + a_{12}a_{21}) + \lambda_1\lambda_3(a_{11}a_{23} + a_{13}a_{21}) + \lambda_2\lambda_3(a_{12}a_{23} + a_{13}a_{22}) \\ &\quad + \lambda_3^2 a_{13}a_{23} + \lambda_1^2 a_{11}a_{21} + \lambda_2^2 a_{12}a_{22}. \end{aligned}$$

The inclusion (1.4.2) is equivalent to the inequality

$$\mathbf{Re}(e^{-i\xi}z) \geq \mathbf{Re}(e^{-i\xi}\lambda_1\lambda_3)$$

for any $(a_{ij}) \in \mathrm{Uni}(3)$. So we shall prove the positivity of $\mathbf{Re}(e^{-i\xi}(z - \lambda_1\lambda_3))$ for any $(a_{ij}) \in \mathrm{Uni}(3)$. This functional depends on the relative positions of $\lambda_i = \mu_i - \mu_0$ for $i = 1, 2, 3$. Firstly we express the coefficients of this functional by using λ_i. Later we express the coefficients by using μ_1, μ_2, μ_3 and μ_0 since the position of the origin to the points λ_1, λ_2 and λ_3 is deeply related with the positivity of the functional.

We shall consider the following normalized functional on Uni(3)

$$L((a_{ij})) = \frac{\mathbf{Re}(e^{-i\xi}(z - \lambda_1\lambda_3))}{\mathbf{Re}(e^{-i\xi}(\lambda_1\lambda_2 - \lambda_1\lambda_3))}.$$

It is rewritten as

$$L((a_{ij})) = a_{11}a_{22} + a_{12}a_{21} + pa_{13}a_{23} + qa_{11}a_{21} + ra_{12}a_{22}, \tag{1.4.3}$$

where

$$p = \frac{\mathbf{Re}(e^{-i\xi}(\lambda_3^2 - \lambda_1\lambda_3))}{\mathbf{Re}(e^{-i\xi}(\lambda_1\lambda_2 - \lambda_1\lambda_3))},$$

$$q = \frac{\mathbf{Re}(e^{-i\xi}(\lambda_1^2 - \lambda_1\lambda_3))}{\mathbf{Re}(e^{-i\xi}(\lambda_1\lambda_2 - \lambda_1\lambda_3))},$$

$$r = \frac{\mathbf{Re}(e^{-i\xi}(\lambda_2^2 - \lambda_1\lambda_3))}{\mathbf{Re}(e^{-i\xi}(\lambda_1\lambda_2 - \lambda_1\lambda_3))}.$$

We may assume $p > 0$ by Lemma 1.3 provided that the point μ_0 belongs to the open disc bounded by the circle circumscribed to $\triangle\mu_1\mu_2\mu_3$.

The positive semi-definiteness of $L((a_{ij}))$ is characterized by the following.

Theorem 1.8. *Let $L((a_{ij}))$ be the functional on* Uni(3) *defined by* (1.4.3). *Then $L((a_{ij})) \geq 0$ for all $(a_{ij}) \in$* Uni(3) *if and only if one of the following conditions holds*:

(i) $0 < p \leq 1$, $p + q \geq 0$ *and* $p + r \geq 0$.

(ii) $p > 1$, $p + q \geq 0$, $p + r \geq 0$ *and* $-1 + 2p + pq + pr + qr \geq 0$.

The corresponding regions of (i) *and* (ii) *in the* (q, r)*-plane are shown in* Figure 2 *and* Figure 3*, respectively.*

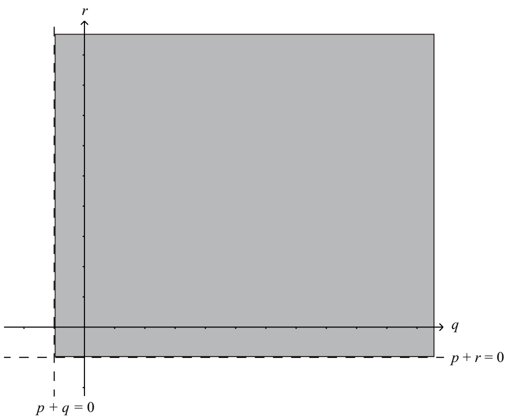

Figure 2

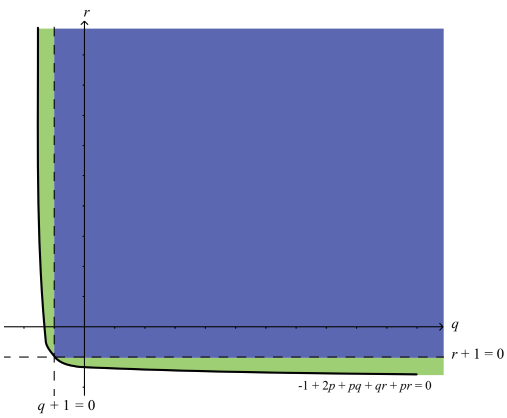

Figure 3

Proof. " If " part: For $0 < p \leq 1$, $p + q \geq 0$ and $p + r \geq 0$, we use

$$\sqrt{a_{13}a_{23}} \geq |\sqrt{a_{11}a_{21}} - \sqrt{a_{12}a_{22}}| \tag{1.4.4}$$

in Theorem 1.1, then

$$\begin{aligned} L((a_{ij})) &\geq pa_{11}a_{22}+pa_{12}a_{21}+pa_{13}a_{23}-pa_{11}a_{21}-pa_{12}a_{22} \\ &\geq pa_{11}a_{22}+pa_{12}a_{21}-2p\sqrt{a_{11}a_{22}}\sqrt{a_{12}a_{21}} \\ &= p(\sqrt{a_{11}a_{22}}-\sqrt{a_{12}a_{21}})^2 \geq 0. \end{aligned}$$

For $p>1$, $p+q\geq 0$, $p+r\geq 0$ and $-1+2p+pq+pr+qr\geq 0$, let $P=p-1$, $Q=q+1$, $R=r+1$ and use (1.4.4) again, we have

$$\begin{aligned} L((a_{ij})) &\geq a_{11}a_{22}+a_{12}a_{21}+Pa_{13}a_{23}+Qa_{11}a_{21}+Ra_{12}a_{22}-2\sqrt{a_{11}a_{12}a_{21}a_{22}} \\ &= (\sqrt{a_{11}a_{22}}-\sqrt{a_{12}a_{21}})^2+Pa_{13}a_{23}+Qa_{11}a_{21}+Ra_{12}a_{22}, \end{aligned}$$

so it is sufficient to show the inequality

$$Pa_{13}a_{23}+Qa_{11}a_{21}+Ra_{12}a_{22}\geq 0 \qquad (1.4.5)$$

for any $(a_{ij})\in \mathrm{Uni}(3)$. If $Q\geq 0$ and $R\geq 0$ hold, the inequality (1.4.5) is trivial. So we may assume that $R>0$ and $Q<0$. The case $Q>0$ and $R<0$ can be treated similarly. The negative constant Q satisfies

$$0<(P+R)(-Q)\leq PR,$$

and hence there exists $Q_1\leq Q$ for which $-Q_1=PR/(P+R)$. It follows from the inequality

$$\sqrt{a_{11}a_{21}}\geq |\sqrt{a_{13}a_{23}}-\sqrt{a_{12}a_{22}}|$$

in Theorem 1.1 that this Q_1 satisfies

$$\begin{aligned} & Pa_{13}a_{23}+Q_1a_{11}a_{21}+Ra_{12}a_{22} \\ \geq\ & Pa_{13}a_{23}+Ra_{12}a_{22}-\tfrac{PR}{P+R}(a_{13}a_{23}+a_{12}a_{22}+2\sqrt{a_{12}a_{13}a_{22}a_{23}}) \\ =\ & \tfrac{1}{P+R}(P^2a_{13}a_{23}+R^2a_{12}a_{22}-2PR\sqrt{a_{12}a_{13}a_{22}a_{23}}) \\ =\ & \tfrac{1}{P+R}(P\sqrt{a_{13}a_{23}}-R\sqrt{a_{12}a_{22}})^2 \\ \geq\ & 0. \end{aligned}$$

Therefore

$$Pa_{13}a_{23}+Qa_{11}a_{21}+Ra_{12}a_{22}\geq Pa_{13}a_{23}+Q_1a_{11}a_{21}+Ra_{12}a_{22}\geq 0$$

for every $(a_{ij})\in \mathrm{Uni}(3)$. This establishes the " If " part.
" Only if " part: For $p>0$ and $L((a_{ij}))\geq 0$ for every $(a_{ij})\in \mathrm{Uni}(3)$, by choosing

$$(a_{ij})=\begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$(a_{ij}) = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \end{bmatrix},$$

we have $p+q \geq 0$ and $p+r \geq 0$, respectively. So it remains to show that $-1+2p+pq+qr+pr \geq 0$ under the condition $p > 1$. Suppose to the contrary that

$$q+1 < \frac{-(p-1)(r+1)}{p+r}.$$

We are to show that $L((a_{ij})) < 0$ for some $(a_{ij}) \in \mathrm{Uni}(3)$. Suppose $q < q_0 < 0$ satisfies

$$q_0+1 = \frac{-(p-1)(r+1)}{p+r}.$$

We note that if q^* satisfies $q \leq q^* < q_0$, then $L((a_{ij})) \leq \tilde{L}((a_{ij}))$ for all $(a_{ij}) \in \mathrm{Uni}(3)$ where

$$\tilde{L}((a_{ij})) = a_{11}a_{22} + a_{12}a_{21} + pa_{13}a_{23} + q^* a_{11}a_{21} + ra_{12}a_{22}.$$

Thus it suffices to show that there is a q^* where $q \leq q^* < q_0$ such that $\tilde{L}((a_{ij})) < 0$ for some $(a_{ij}) \in \mathrm{Uni}(3)$. We express an element (a_{ij}) of $\partial\mathrm{Uni}(3)$ as

$$a_{13} = t^2,\ a_{12} = \frac{1}{2}(1-t^2)\frac{2w^2}{1+w^2},\ a_{11} = \frac{1}{2}(1-t^2)\frac{2}{1+w^2},\ a_{23} = \frac{1}{2}(1-t^2)\frac{2u^2}{1+u^2},$$

$$a_{22} = \frac{1}{(1+u^2)(1+w^2)}(tuw-1)^2,\ a_{21} = \frac{1}{(1+u^2)(1+w^2)}(tu+w)^2,$$

for $-1 \leq t \leq 1$, $-\infty < u, w < \infty$, by using three Euler angles of the rotation group $SO(3)$ and rational parametrizations of $\cos\phi$, $\sin\phi$ and $\cos\psi$, $\sin\psi$. Next, we consider p and r are fixed and q is closed to q_0. Then $L((a_{ij})) = L(t,u,w,q)$ and

$$L = \frac{1-t^2}{(1+u^2)(1+w^2)^2} M_1,$$

where

$$\begin{aligned} M_1 &= u^2(pt^2 + qt^2 + 2t^2w^2 + 2pt^2w^2 + pt^2w^4 + rt^2w^4) \\ &\quad + u(-2tw + 2qtw + 2tw^3 - 2rtw^3) + 1 + qw^2 + rw^2 + w^4 \\ &= (tu)^2(p + q + 2w^2 + 2pw^2 + pw^4 + rw^4) \\ &\quad + (tu)(-2w + 2qw + 2w^3 - 2rw^3) + (1 + qw^2 + rw^2 + w^4). \end{aligned}$$

Regarding this as a quadratic polynomial in tu, its minimum value is attained at any values of t and u satisfying

$$tu = g(w,q) = \frac{w(rw^2 - w^2 - q + 1)}{(p+r)w^4 + 2(p+1)w^2 + p + q}, \tag{1.4.6}$$

and the minimum value is

$$M_1|_{tu=g(w,q)} = (1 + qw^2 + rw^2 + w^4) - \frac{(w - qw - w^3 + rw^3)^2}{(p+r)w^4 + 2(p+1)w^2 + p + q}.$$

Then

$$L|_{tu=g(w,q)} = \frac{1-t^2}{(1+u^2)(1+w^2)^2} M_1|_{tu=g(w,q)}$$

and, by direct calculations,

$$\frac{M_1|_{tu=g(w,q)}}{(1+w^2)^2} = \frac{(p+r)w^4 + (1 + pq + pr + qr)w^2 + p + q}{(p+r)w^4 + 2(p+1)w^2 + p + q}.$$

Since $(1-t^2)/(1+u^2) > 0$, to show that L can take a negative value, it suffices to show that

$$(p+r)w^4 + (1 + pq + pr + qr)w^2 + p + q$$

can take a negative value. Regarding it as a quadratic polynomial in w^2, its minimum value is attained at $w = h(q)$ where

$$h(q)^2 = \frac{-(1 + pq + pr + qr)}{2(p+r)}$$

and the minimum value is

$$p + q - \frac{(1 + pq + qr + pr)^2}{4(p+r)}.$$

Note that as $(p+r)q_0 = 1 - 2p - pr$, we have $1 + pq_0 + pr + q_0 r = 2 - 2p < 0$. With q sufficiently closed to q_0, the $h(q)$ above is valid. Let

$$f(q) = p + q - \frac{(1 + pq + qr + pr)^2}{4(p+r)}.$$

It is easy to check that $f(q_0) = 0$ and $f'(q_0) = p > 0$. Thus we may choose some $q^* < q_0$ sufficiently closed to q_0 such that $f(q^*) < 0$ and $w^* = h(q^*)$ is valid. Let u^* and v^* be chosen such that $t^* u^* = g(w^*, q^*)$. Thus there is a point $(a_{ij}) \in \partial\mathrm{Uni}(3)$ with $L((a_{ij})) = L(t^*, u^*, w^*, q^*) < 0$ as required.

$\square$

Next we shall examine the geometric meaning for the points μ_j ($j = 1,2,3$) and μ_0 of the following six inequalities

$$\begin{array}{lll} (C_1): p+q \geq 0, & (C_2): p+r \geq 0, & (C_3): -1+2p+pq+pr+qr \geq 0, \\ (C_4): p-1 \leq 0, & (C_5): q+1 \geq 0, & (C_6): r+1 \geq 0. \end{array}$$

In the statement of Theorem 1.8, the four inequalities (C_1), (C_2), (C_3) and (C_4) in the above list appear. In Figure 3 for $p > 1$, the area for (C_3) is the union of the three regions:

(ii-1) $q+1 \geq 0$ and $r+1 \geq 0$,

(ii-2) $q+1 > 0$ and $0 > r+1 > \dfrac{-(p-1)(q+1)}{p+q} > -p+1$,

(ii-3) $r+1 > 0$ and $0 > q+1 > \dfrac{-(p-1)(r+1)}{p+r} > -p+1$.

In these statements, the conditions (C_5) and (C_6) of the above list appear. We remark that if the conditions $(C_1), (C_2), (C_5), (C_6)$ and $p > 1$ are satisfied, then the condition (C_3) is satisfied by the following reason:

$$-1+2p+pq+pr+qr = (p-1)(r+1)+(q+1)(p+r) \geq 0.$$

The region (or boundary) corresponding to (C_j) is denoted by D_j.

We shall reduce the proof of Theorem 1.4 for a general triangle $\triangle\mu_1\mu_2\mu_3$ to a special type triangle in the case

$$\mu_1 = iy_1, \quad \mu_2 = iy_2, \quad \mu_3 = a. \tag{1.4.7}$$

for some $y_2 \leq 0 \leq y_1$ and $a > 0$. We deduce such a reduction. In the general setting,

$$\begin{aligned} \mu_1 - \mu_2 &= re^{i\rho}(e^{i(2\eta_1+2\eta_2)} - 1) \\ &= 2ir\sin(\eta_1+\eta_2)e^{i(\eta_1+\eta_2+\rho)}. \end{aligned}$$

So

$$\arg(\mu_1 - \mu_2) \equiv \eta_1 + \eta_2 + \rho + \frac{\pi}{2} \mod 2\pi.$$

By using a rotation, we may assume that

$$\eta_1 + \eta_2 + \rho \equiv 0 \mod 2\pi,$$

and hence

$$\arg(\mu_1 - \mu_2) \equiv \frac{\pi}{2} \mod 2\pi$$

and

$$\mathbf{Re}(\mu_1) = \mathbf{Re}(\mu_2) < \mathbf{Re}(\mu_3).$$

We set

$$\begin{array}{rcl} a & = & \mathbf{Re}(\mu_3) - \mathbf{Re}(\mu_1) > 0 \\ y_1 & = & \mathbf{Im}(\mu_1) - \mathbf{Im}(\mu_3) \\ y_2 & = & \mathbf{Im}(\mu_2) - \mathbf{Im}(\mu_3). \end{array}$$

Since $\arg(\mu_1 - \mu_2) \equiv \frac{\pi}{2} \bmod 2\pi$, the inequality $y_1 > y_2$ holds. The triangle $\triangle\mu_2\mu_3\mu_1$ is acute-angled or right-angled and hence $y_1 \geq 0 \geq y_2$. By using a translation, we may assume that

$$\mathbf{Re}(\mu_1) = \mathbf{Re}(\mu_2) = 0 \text{ and } \mathbf{Im}(\mu_3) = 0$$

and hence we have (1.4.7).

Our argument to prove Theorem 1.4 will depend on a number of observations. Before our argument, we shall consider the conditions for μ_0 to satisfy (1.4.2) where

$$\lambda_j = \mu_j - \mu_0 \text{ for } j = 1, 2, 3.$$

We apply the assumption (1.4.7). Since $\lambda_1 - \lambda_2 = \mu_1 - \mu_2$ is pure imaginary, we multiply $\lambda_j\lambda_k$ and λ_k^2 by $e^{-i\xi} = \overline{\lambda_3}/|\lambda_3|$.

Let $\mu_0 = \alpha + i\beta$. With $e^{-i\xi} = \overline{\lambda_3}/|\lambda_3|$,

$$\begin{aligned} p &= \frac{\mathbf{Re}(e^{-i\xi}(\lambda_3^2 - \lambda_1\lambda_3))}{\mathbf{Re}(e^{-i\xi}(\lambda_1\lambda_2 - \lambda_1\lambda_3))} \\ &= \frac{\mathbf{Re}(\lambda_3 - \lambda_1)}{\mathbf{Re}((\lambda_1(\lambda_2 - \lambda_3)\overline{\lambda_3})/(\lambda_3\overline{\lambda_3}))} \\ &= \frac{\mathbf{Re}(\lambda_3 - \lambda_1)|\lambda_3|^2}{\mathbf{Re}(\lambda_1(\lambda_2 - \lambda_3)\overline{\lambda_3})} \\ &= \frac{a((a-\alpha^2) + \beta^2)}{\mathbf{Re}\left[(-\alpha + i(y_1 - \beta))(-a + iy_2)((a-\alpha) + i\beta)\right]} \\ &= \frac{a((a-\alpha)^2 + \beta^2)}{C(\alpha, \beta)}, \end{aligned} \tag{1.4.8}$$

and the expressions for q and r can be obtained:

$$q = \frac{-a\alpha^2 - a\beta^2 + a^2\alpha + y_1^2\alpha + 2ay_1\beta - ay_1^2}{C(\alpha, \beta)}, \tag{1.4.9}$$

and

$$r = \frac{-a\alpha^2 - a\beta^2 + a^2\alpha + y_2^2\alpha + 2ay_2\beta - ay_2^2}{C(\alpha, \beta)}, \tag{1.4.10}$$

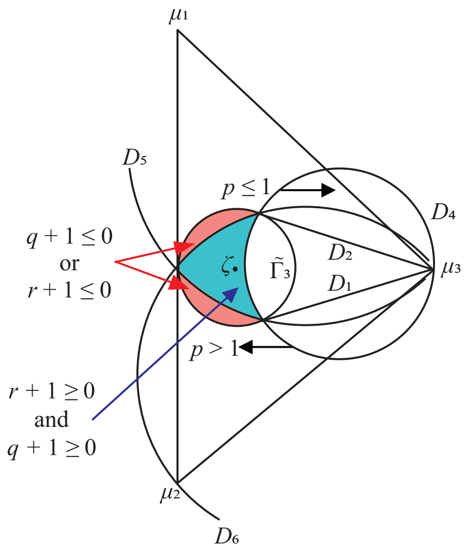

Figure 4. ζ is the circumcenter of $\triangle\mu_1\mu_2\mu_3$.

where

$$C(x, y) = -ax^2 - ay^2 + a^2x + y_1y_2x + ay_1y + ay_2y - ay_1y_2 = 0$$

is the circumscribing circle of $\triangle\mu_1\mu_2\mu_3$. Its center $x_0 + iy_0$ is given by

$$x_0 = \frac{a^2 + y_1y_2}{2a} \text{ and } y_0 = \frac{y_1 + y_2}{2}.$$

Since $\mu_0 = (\alpha, \beta)$ is inside this circumscribing circle, we have $C(\alpha, \beta) > 0$. By using these expressions, the six conditions (C_j) on the point μ_0 are explained by four circular discs and two half-planes.

Observation 1. The condition (C_3) for $\mu_0 = (\alpha, \beta)$ is satisfied if and only if μ_0 belongs the circular disc $\tilde{\Gamma}_3$ in Theorem 1.4

$$\tilde{\Gamma}_3 : (x - x_0)^2 + (y - y_0)^2 \le x_0^2.$$

Proof. We shall express the condition (C_3) for $\mu_0 = \alpha + i\beta$ as some inequality on $(\alpha, \beta) \in \mathbb{R}^2$. For this purpose, we substitute the relations (1.4.8), (1.4.9) and (1.4.10) into the inequality

$$(C_3) : -1 + 2p + pq + pr + qr \ge 0.$$

Then we get a rational function $P(\alpha,\beta)/C(\alpha,\beta)^2$ where the polynomial $P(\alpha,\beta)$ is given by

$$\begin{aligned}
P(\alpha,\beta) &= -(-a\alpha^2 - a\beta^2 + a^2\alpha + y_1y_2\alpha + ay_1\beta + ay_2\beta - ay_1y_2)^2 \\
&\quad +2a((\alpha-a)^2+\beta^2)(-a\alpha^2 - a\beta^2 + a^2\alpha + y_1y_2\alpha + ay_1y + ay_2\beta - ay_1y_2) \\
&\quad +a((\alpha-a)^2+\beta^2)(-a\alpha^2 - a\beta^2 + a^2\alpha + y_1^2\alpha + 2ay_1\beta - ay_1^2) \\
&\quad +a((\alpha-a)^2+\beta^2)(-a\alpha^2 - a\beta^2 + a^2\alpha + y_2^2x + 2ay_2y - ay_2^2) \\
&\quad +(-a\alpha^2 - a\beta^2 + a^2\alpha + y_1^2\alpha + 2ay_1\beta - ay_1^2)(-a\alpha^2 - a\beta^2 \\
&\qquad\qquad +a^2\alpha + y_2^2\alpha + 2ay_2\beta - ay_2^2) \\
&= a((a-\alpha)^2+\beta^2)(-4a\alpha^2 - 4ay^2 + 4a^2\alpha + 4y_1y_2\alpha \\
&\qquad\qquad +4a(y_1+y_2)\beta - a(y_1+y_2)^2) \\
&= 4a^2((a-\alpha)^2+\beta^2)(-(\alpha-x_0)^2 - (\beta-y_0)^2 + x_0^2).
\end{aligned}$$

We shall determine the domain corresponding to the condition (C_3) under the condition $p > 0$ and hence $C(\alpha,\beta) > 0$. The point $(\alpha,\beta) = (a,0)$ satisfies $C(\alpha,\beta) = 0$. So we disregard this point. By removing the point $(\alpha,\beta) = (a,0)$ the domain $P(\alpha,\beta) \geq 0$ for the condition is expressed as the circular disc $\tilde{\Gamma}_3$. □

The circumference of $\tilde{\Gamma}_3$ has the tangent $x = 0$ at the midpoint of the line segment $[\mu_1,\mu_2]$. The point $\mu_3 = a$ does not belong to $\tilde{\Gamma}_3$.

Observation 2. The condition (C_4) on $\mu_0 = (\alpha,\beta)$ is satisfied if and only if μ_0 belongs to the circular disc

$$D_4 : (x - \frac{x_0+a}{2})^2 + (y - \frac{y_0}{2})^2 \leq \frac{(a-x_0)^2+y_0^2}{4}.$$

Proof. We substitute (1.4.8) into the condition (C_4) and express $C(\alpha,\beta)(p-1)$ as a polynomial of α and β:

$$\begin{aligned}
C(\alpha,\beta)(p-1) &= a^3 - 3a^2\alpha + 2a\alpha^2 + 2a\beta^2 - a\beta y_1 - a\beta y_2 + ay_1y_2 - \alpha y_1y_2 \\
&= 2a\left[(\alpha - \frac{x_0+a}{2})^2 + (\beta - \frac{y_0}{2})^2 - \frac{(a-x_0)^2+y_0^2}{4}\right].
\end{aligned}$$

Thus we obtained the region of $\mu_0 = \alpha + i\beta$ for the condition (C_4) as the above statement. This region contains the point $\mu_3 = a$ at which the coefficient p vanishes. □

The line segment $[x_0 + iy_0, \mu_3]$ is the diameter of D_4.

Observation 3. The two conditions (C_1), (C_2) on $\mu_0 = (\alpha, \beta)$ are satisfied if and only if

$$\frac{(a^2 - y_1^2)\mathbf{Re}(\mu_0) - a^3 + ay_1^2}{2ay_1} \leq \mathbf{Im}(\mu_0) \leq \frac{(a^2 - y_2^2)\mathbf{Re}(\mu_0) - a^3 + ay_2^2}{2ay_2}, \tag{1.4.11}$$

holds.

Proof. By using (1.4.8), (1.4.9) and (1.4.10), we express the conditions (C_1) and (C_2) as some inequalities of (α, β). The circumferences of the circular discs $\tilde{\Gamma_3}$ and D_4 intersect at two points on the line

$$(2a^2 - 2y_1y_2)x - 2a(y_1 + y_2)y - 2a^3 + a(y_1^2 + y_2^2) = 0.$$

The two intersection points of the two circles are tangent points of the tangent lines to $\tilde{\Gamma_3}$ from the point μ_3. The two tangent lines are given by

$$D_1 : y = \frac{a^2 - y_1^2}{2ay_1}(x - a)$$

and

$$D_2 : y = \frac{a^2 - y_2^2}{2ay_2}(x - a)$$

in the case $y_2 < 0 < y_1$. In the strip

$$\frac{a^2 + y_1y_2 - \sqrt{(a^2 + y_1^2)(a^2 + y_2^2)}}{2a} \leq x \leq a, \tag{1.4.12}$$

the two conditions (C_1), (C_2) are satisfied if and only if y satisfies (1.4.11). We remark that the above inequality defines a non-empty set by the assumption $a^2 + y_1y_2 \geq 0$. The intersection of the range D_4 and the range (1.4.11) is one part of the area in which μ_0 satisfies $L((a_{ij})) \geq 0$ for any $(a_{ij}) \in \mathrm{Uni}(3)$. In the case $y_1 = 0$ or $y_2 = 0$, one tangent line to the first circular disc is a vertical line $x = a$. □

We remark that the region satisfying the conditions (C_1) and (C_2) is empty under the setting $y_1 > 0$, $y_2 < 0$ and $a^2 + y_1y_2 < 0$, that is the angle $\angle\mu_2\mu_3\mu_1$ is obtuse. In fact, the set of points μ_0 in the above strip (1.4.12) satisfying the inequality (1.4.11) is empty.

Observation 4. The conditions (C_5) and (C_6) on $\mu_0 = \alpha + i\beta$ are satisfied if and only if (α, β) belongs to the respective circular discs

$$D_5 : x^2 + y^2 - (a + \frac{y_1(y_1 + y_2)}{2a})x - \frac{3y_1 + y_2}{2}y + \frac{y_1(y_1 + y_2)}{2} \leq 0 \tag{1.4.13}$$

and

$$D_6 : x^2 + y^2 - (a + \frac{y_2(y_1 + y_2)}{2a})x - \frac{3y_2 + y_1}{2}y + \frac{y_2(y_1 + y_2)}{2} \leq 0. \quad (1.4.14)$$

Proof. Similarly, we substitute (1.4.9) and (1.4.10) into $C(\alpha, \beta)(q + 1)$ and $C(\alpha, \beta)(r + 1)$, respectively. So we have

$$\begin{aligned} C(\alpha,\beta)(q+1) &= 2a^2\alpha - 2a\alpha^2 - 2a\beta^2 + 3a\beta y_1 - ay_1^2 + \alpha y_1^2 + a\beta y_2 - ay_1y_2 + \alpha y_1 y_2 \\ &= -2a\left[\alpha^2 + \beta^2 - (a + \frac{y_1(y_1 + y_2)}{2a})\alpha - \frac{3y_1 + y_2}{2}\beta + \frac{y_1(y_1 + y_2)}{2}\right] \end{aligned}$$

and

$$\begin{aligned} C(\alpha,\beta)(r+1) &= 2a^2\alpha - 2a\alpha^2 - 2a\beta^2 + 3a\beta y_2 - ay_2^2 + \alpha y_2^2 + a\beta y_1 - ay_1y_2 + \alpha y_1 y_2 \\ &= -2a\left[\alpha^2 + \beta^2 - (a + \frac{y_2(y_1 + y_2)}{2a})\alpha - \frac{3y_2 + y_1}{2}\beta + \frac{y_2(y_1 + y_2)}{2}\right]. \end{aligned}$$

Thus the conditions (C_5) and (C_6) on $\mu_0 = \alpha + i\beta$ are satisfied if and only if (α, β) belongs to (1.4.13) and (1.4.14), respectively. □

In Figure 4, their boundary curves D_5 and D_6 are also shown. The intersection point of D_1 and D_5 other than $\mu_3 = a$ is given by

$$(x, y) = \left(\frac{a^3 + ay_1y_2}{a^2 + y_1^2}, -\frac{(y_1 - y_2)(a^2 - y_1^2)}{2(a^2 + y_1^2)}\right).$$

Similarly, the intersection point of D_2 and D_6 is given by

$$(x, y) = \left(\frac{a^3 + ay_1y_2}{a^2 + y_2^2}, -\frac{(y_2 - y_1)(a^2 - y_2^2)}{2(a^2 + y_2^2)}\right).$$

At these two points, the circles D_3 and D_4 intersect. We consider the convex hull Γ_3 of the circular disc $\tilde{\Gamma}_3$ and the point μ_3. By summing up the above four observations, we conclude that a point $\mu_0 = \alpha + i\beta$ with $C(\alpha, \beta) > 0$ satisfies

$$L((a_{ij})) = a_{11}a_{22} + a_{12}a_{21} + pa_{13}a_{23} + qa_{11}a_{21} + ra_{12}a_{22} \geq 0$$

for any $(a_{ij}) \in \mathrm{Uni}(3)$ and the coefficients p, q, r corresponding to

$$(\lambda_1, \lambda_2, \lambda_3) = (\mu_1 - \mu_0, \mu_2 - \mu_0, \mu_3 - \mu_0)$$

if and only if $\mu_0 \in \Gamma_3$.

Similarly we consider the functional

$$L_1((a_{ij})) = a_{12}a_{23} + a_{13}a_{22} + p_1a_{11}a_{21} + q_1a_{12}a_{22} + r_1a_{13}a_{23}$$

for $(a_{ij}) \in \mathrm{Uni}(3)$ and the coefficients p_1, q_1, r_1 corresponding to $\lambda_j = \mu_j - \mu_0$ for $j = 1, 2, 3$ and the functional

$$L_2((a_{ij})) = a_{11}a_{23} + a_{13}a_{21} + p_2a_{12}a_{22} + q_2a_{13}a_{23} + r_2a_{11}a_{21}$$

for $(a_{ij}) \in \mathrm{Uni}(3)$ and the coefficients p_2, q_2, r_2 corresponding to $\lambda_j = \mu_j - \mu_0$ for $j = 1, 2, 3$. We consider the disc $\tilde{\Gamma_1}$ with center at the circumcenter ζ of $\triangle\mu_1\mu_2\mu_3$ with the radius equal to the distance of ζ from the line segment $[\mu_2, \mu_3]$. The disc $\tilde{\Gamma_2}$ has the center ζ and the radius equal to the distance of ζ from the line segment $[\mu_1, \mu_3]$. We define

$$\Gamma_1 = \mathrm{Conv}(\tilde{\Gamma_1}, \mu_1) \text{ and } \Gamma_2 = \mathrm{Conv}(\tilde{\Gamma_2}, \mu_2).$$

Then the condition

$$(\lambda_1a_{11} + \lambda_2a_{12} + \lambda_3a_{13})(\lambda_1a_{21} + \lambda_2a_{22} + \lambda_3a_{23}) \in \mathrm{Conv}(\lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3)$$

holds for every $(a_{ij}) \in \mathrm{Uni}(3)$ for the triple

$$(\lambda_1, \lambda_2, \lambda_3) = (\mu_1 - \mu_0, \mu_2 - \mu_0, \mu_3 - \mu_0)$$

if and only if

$$\mu_0 \in \Gamma_1 \cap \Gamma_2 \cap \Gamma_3,$$

where we assume that

$$0 < \arg\left(\frac{\mu_3 - \mu_1}{\mu_2 - \mu_1}\right), \arg\left(\frac{\mu_1 - \mu_2}{\mu_3 - \mu_2}\right), \arg\left(\frac{\mu_2 - \mu_3}{\mu_1 - \mu_3}\right) \leq \frac{\pi}{2}.$$

and

$$|\mu_0 - \zeta| = |\mu_0 - (x_0 + iy_0)| < r = \frac{\sqrt{(a^2 + y_1^2)(a^2 + y_2^2)}}{2a}.$$

We now complete the proof of Theorem 1.4.

Proof of Theorem 1.4. We shall prove the converse inclusion of (1.4.1) under the condition that $|\mu_0 - \zeta| < r$ and $\mu_0 \in \Gamma_j$ for $j = 1, 2, 3$. Then the above arguments imply

$$\left(\frac{\lambda_1 + \lambda_2 + \lambda_3}{3}\right)^2 \in W_2^{\Pi}(\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)) \subset \mathrm{Conv}(\lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3). \tag{1.4.15}$$

Let

$$A_{j,k} = \left\{ \left[s\lambda_j + (1-s)(\frac{\lambda_1+\lambda_2+\lambda_3}{3}) \right] \left[s\lambda_k + (1-s)(\frac{\lambda_1+\lambda_2+\lambda_3}{3}) \right] : 0 \le s \le 1 \right\}$$

be the arc between $\lambda_j\lambda_k$ and $(\lambda_1+\lambda_2+\lambda_3)^2/9$ for $1 \le j < k \le 3$. Then the arc $A_{j,k}$ is the image of $sC_0 + (1-s)P_{j,k}$ under the map

$$(a_{ij}) \mapsto (\lambda_1 a_{11} + \lambda_2 a_{12} + \lambda_3 a_{13})(\lambda_1 a_{21} + \lambda_2 a_{22} + \lambda_3 a_{23})$$

where C_0 is the Van der Waerden matrix and $P_{j,k}$ is an odd permutation matrix. By Theorem 1.2, we have $sC_0 + (1-s)P_{j,k} \in \mathrm{Uni}(3)$. Thus the above relation (1.4.15) implies that the 3 arcs $A_{j,k}$ for $1 \le j < k \le 3$ are contained in the triangle $\mathrm{Conv}(\lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3)$ provided that $\mu_0 \in \Gamma_1 \cap \Gamma_2 \cap \Gamma_3$. The

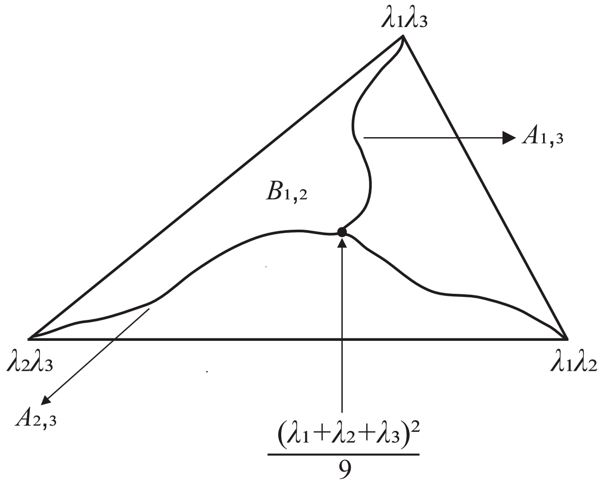

Figure 5. $\mathrm{Conv}(\lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3)$

range

$$B_{1,2} = \left\{ \left[s(t\lambda_1 + (1-t)\lambda_2) + (1-s)(\frac{\lambda_1+\lambda_2+\lambda_3}{3}) \right] \left[s\lambda_3 + (1-s)(\frac{\lambda_1+\lambda_2+\lambda_3}{3}) \right] : 0 \le s, t \le 1 \right\}$$

is also contained in the triangle $\mathrm{Conv}(\lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3)$ since the matrix

$$s \begin{bmatrix} t & 1-t & 0 \\ 0 & 0 & 1 \\ 1-t & t & 0 \end{bmatrix} + (1-s) \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \in \mathrm{Uni}(3)$$

for any $0 \le s, t \le 1$, by Theorem 1.2. The range $B_{1,2}$ consists of line segments

$$\left[s(t\lambda_1 + (1-t)\lambda_2) + (1-s)(\frac{\lambda_1+\lambda_2+\lambda_3}{3}) \right] \left[s\lambda_3 + (1-s)(\frac{\lambda_1+\lambda_2+\lambda_3}{3}) \right]$$

for fixed s. We also consider similar ranges $B_{1,3}$ and $B_{2,3}$. Then $\mathrm{Conv}(\lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3)$ is expressed as the union of three ranges $B_{j,k}$ $(j < k)$. So the range $W_2^{\Pi}(\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3))$ contains the triangle $\mathrm{Conv}(\lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3)$.

We shall prove that the inclusion (1.4.1) holds only if the point μ_0 belongs to $\Gamma_1 \cap \Gamma_2 \cap \Gamma_3$. If $|\mu_0 - \zeta| < r$ with $\mu_0 \notin \Gamma_3$, then the geometric consideration on the coefficients p, q, r of the functional $L((a_{ij}))$ imply that

$$-1 + 2p + pq + pr + rq < 0$$

and $p > 1$. By Theorem 1.8, $L((a_{ij}))$ has a negative value for some $(a_{ij}) \in \mathrm{Uni}(3)$. That is, there is a point of the form

$$z = \prod_{j=1}^{2} (\lambda_1 a_{j1} + \lambda_2 a_{j2} + \lambda_3 a_{j3})$$

that does not belong to the closed half plane H_{12} with the boundary line passing through $\lambda_1\lambda_2, \lambda_2\lambda_3$ and $\lambda_1\lambda_3 \in H_{12}$. This means

$$z \in W_2^{\Pi}(\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3))$$

but

$$z \notin \mathrm{Conv}(\lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3)$$

and hence μ_0 does not satisfy (1.4.1). □

Next we shall complete the proof of Corollary 1.6 by proving that $\mu_0 \in \Gamma_1 \cap \Gamma_2 \cap \Gamma_3$.

Proof of Corollary 1.6. We prove μ_0 belongs to Γ_3. By similar arguments, we can prove that μ_0 belongs $\Gamma_1 \cap \Gamma_2$. By using the setting

$$\mu_1 = iy_1, \quad \mu_2 = iy_2, \quad \mu_3 = a,$$

the point $\mu_0 = (\mu_1 + \mu_2 + \mu_3)/3$ is expressed as

$$\alpha = \frac{a}{3} \text{ and } \beta = \frac{y_1 + y_2}{3}.$$

For this point, $p + q$, $p + r$ and p are expressed as

$$p = \frac{4a^2 + (y_1 + y_2)^2}{2(a^2 + y_1^2 - y_1 y_2 + y_2^2)}$$

and

$$p+q=p+r=\frac{3(a^2+y_1y_2)}{a^2+y_1^2-y_1y_2+y_2^2}.$$

Hence p, $p+q$ and $p+r$ are nonnegative. For $p \leq 1$, by Theorem 1.8 (i) and the arguments in the proof of Theorem 1.4, we can conclude that $\mu_0 \in \Gamma_3$. For $p > 1$, the condition $-1+2p+pq+qr+pr$ is expressed as

$$-1+2p+pq+qr+pr=\frac{32a^4-(y_1+y_2)^2(y_1^2-10y_1y_2+y_2^2)+4a^2(y_1^2+14y_1y_2+y_2^2)}{4(a^2+y_1^2-y_1y_2+y_2^2)^2}. \tag{1.4.16}$$

Since

$$p-1=\frac{2a^2-y_1^2+4y_1y_2-y_2^2}{2(a^2+y_1^2-y_1y_2+y_2^2)}>0,$$

we substitute

$$a^2=\frac{y_1^2-4y_1y_2+y_2^2+\varepsilon^2}{2}$$

into (1.4.16) and get

$$-1+2p+pq+qr+pr=8\varepsilon^4+18\varepsilon^2(y_1-y_2)^2+9(y_1-y_2)^4\geq 0.$$

By Theorem 1.8 (ii) and the arguments in the proof of Theorem 1.4, μ_0 belongs Γ_3, which completes the proof of Corollary 1.6. □

1.5 Obtuse-angled triangle

In this section we consider some possibility of generalizations of Corollary 1.5 and Corollary 1.6.

Proposition 1.9. *Let λ_1, λ_2 and λ_3 be three distinct points on the unit circle $|z|=1$. Then the equation*

$$W_2^{\Pi}(\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3))=\mathrm{Conv}(\lambda_1\lambda_2,\lambda_2\lambda_3,\lambda_1\lambda_3), \tag{1.5.1}$$

holds only when $\triangle\lambda_1\lambda_2\lambda_3$ is an acute-angled or a right-angled triangle.

Proof. We consider a general situation of an obtuse-angled triangle for Corollary 1.5. Let

$$\lambda_1=e^{i\omega},\quad \lambda_2=e^{-i\omega},\quad \lambda_3=e^{i\eta}$$

for $0 \leq \eta < \omega < \pi/2$. Then the functional $L((a_{ij}))$ on Uni(3) is given by

$$L((a_{ij}))=a_{11}a_{22}+a_{12}a_{21}+a_{13}a_{23}+\frac{\cos(2\omega-\eta)-\cos\omega}{\cos\eta-\cos\omega}a_{11}a_{21}+\frac{\cos(2\omega+\eta)-\cos\omega}{\cos\eta-\cos\omega}a_{12}a_{22}.$$

The condition $L((a_{ij})) \geq 0$ for $(a_{ij}) \in \mathrm{Uni}(3)$ is necessary for the equation (1.5.1). If the inequality $L((a_{ij})) \geq 0$ holds for any $(a_{ij}) \in \mathrm{Uni}(3)$, then the coefficients

$$p = 1,$$

$$q = \frac{\cos(2\omega - \eta) - \cos\omega}{\cos\eta - \cos\omega} = -\frac{\sin((3\omega - \eta)/2)}{\sin((\omega + \eta)/2)},$$

and

$$r = \frac{\cos(2\omega - \eta) - \cos\omega}{\cos\eta - \cos\omega} = -\frac{\sin((3\omega + \eta)/2)}{\sin((\omega - \eta)/2)}$$

have to satisfy $q + 1 \geq 0$, $r + 1 \geq 0$. These inequalities are rewritten as

$$-\sin((3\omega - \eta)/2) + \sin((\omega + \eta)/2) = -2\cos\omega \sin((\omega - \eta)/2) \geq 0$$

and

$$-\sin((3\omega + \eta)/2) + \sin((\omega - \eta)/2) = -2\cos\omega \sin((\omega + \eta)/2) \geq 0.$$

But these relations are impossible since $0 \leq \eta < \omega < \pi/2$. Thus we can conclude that the triangle $\lambda_1\lambda_2\lambda_3$ is necessary to be acute-angled or right-angled for the equation (1.5.1) to holds provided that 0 is the circumcenter of $\triangle\lambda_1\lambda_2\lambda_3$. □

Proposition 1.10. *Let λ_1, λ_2 and λ_3 be three distinct complex numbers satisfying*

$$\lambda_1 + \lambda_2 + \lambda_3 = 0.$$

Then the equation (1.5.1) holds only when $\triangle\lambda_1\lambda_2\lambda_3$ is an acute-angled or a right-angled triangle.

Proof. By the equation (2.2), we may assume that $\lambda_1, \lambda_2, \lambda_3$ are not colinear. We consider a general situation of an obtuse-angle triangle for Corollary 1.6. By using a rotation and a similarity, we may assume that

$$\lambda_1 = -1 + iy_1, \quad \lambda_2 = -1 - i(y_1 + y_2), \quad \lambda_3 = 2 + iy_2$$

for some $0 \leq y_2 < y_1$ satisfying $(y_1 - y_2)(y_1 + 2y_2) > 9$. The functional $L((a_{ij}))$ on $\mathrm{Uni}(3)$ is given by

$$L((a_{ij})) = a_{11}a_{22} + a_{12}a_{21} + pa_{13}a_{23} + q(a_{11}a_{21} + a_{12}a_{22})$$

for

$$p = \frac{3(4 + y_2^2)}{2(3 + y_1^2 + y_1y_2 + y_2^2)}, \; q = \frac{6 - 2y_1^2 - 2y_1y_2 + y_2^2}{2(3 + y_1^2 + y_1y_2 + y_2^2)}$$

and hence

$$p + q = \frac{9 - (y_1 - y_2)(y_1 + 2y_2)}{3 + y_1^2 + y_1 y_2 + y_2^2} < 0.$$

Thus we can conclude that $\triangle\lambda_1\lambda_2\lambda_3$ is necessary to be acute-angled or right-angled for the equation (1.5.1) to hold provided that 0 is the centroid of $\triangle\lambda_1\lambda_2\lambda_3$. □

1.6 Plotting programs and examples

Finally, we provide two MATLAB programs for plotting the 2-product range and 3-product range of any 3×3 normal matrix. For convenience, they are written as an m-file for MATLAB. The range $W_k^{\Pi}(N)$ is a compact subset of the Guassian plane $\mathbb{C}$. We approximate it by its finite many representative points. We adopt 200^3 as the number of representative points. The following program is in order to plot the 2-product range.

```
function y = p(p,q,r,m)
a = ones(1,m+1);
t = 0:pi/m:pi;
s = 0:pi/m:pi;
u = 0:pi/m:pi;
b1 = kron(a,a);
b11 = kron(cos(t),b1);
b12 = kron(cos(s),a);
b12 = kron(sin(t),b12);
b21 = kron(a,sin(u));
b21 = kron(sin(t),b21);
b2 = kron(cos(s),sin(u));
bb2 = kron(cos(t),b2);
b3 = kron(sin(s),cos(u));
bb3 = kron(a,b3);
b22 = -bb2 - bb3;
a11 = b11.^2;
a12 = b12.^2;
a21 = b21.^2;
a22 = b22.^2;
a13 = 1 - a11 - a12;
a23 = 1 - a21 - a22;
a31 = 1 - a11 - a21;
a32 = 1 - a12 - a22;
```

```
a33 = 1 - a31 - a32;
X = a11 .* p + a12 .* q + a13 .* r;
Y = a21 .* p + a22 .* q + a23 .* r;
Z = X .* Y;
XX = real(Z);
YY = imag(Z);
grid;
plot(XX,YY)
```

For plotting the 3-product range of a 3×3 normal matrix, we only need to replace 5 codes from the bottom to the above by following:

```
Z = a31 .* p + a32 .* q + a33 .* r;
ZZ = X .* Y .* Z;
XX = real(ZZ);
YY = imag(ZZ);
grid;
plot(XX,YY)
```

In the above two programs, p, q and r are the diagonal entries of the given normal matrix N. For finer approximation, we can replace m by another large number. Of course, it needs more computational times for larger number m.

Next we will provide some examples by using the above codes. We set $m = 200$ in the programs.

Example 1.11. Let $N = \operatorname{diag}(1, \omega, \omega^2)$ where $\omega^3 = 1$. Figure 1 and Figure 2 are the graphs of $W_2^\Pi(N)$ and $W_3^\Pi(N)$, respectively. Note that the

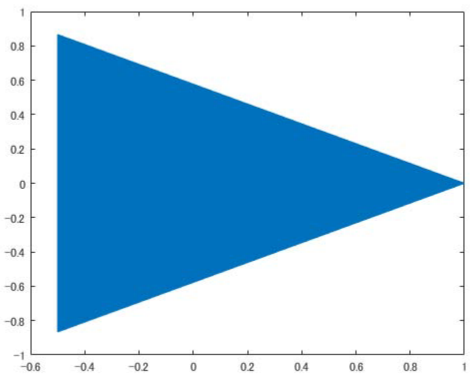

Figure 1: $W_2^\Pi(N)$

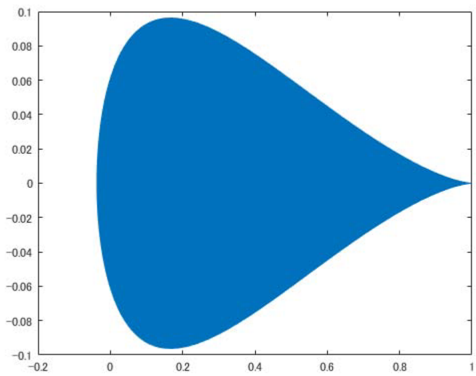

Figure 2: $W_3^\Pi(N)$

diagonal entries of N satisfy Theorem 2.1 in [29], so $W_2^\Pi(N)$ is a triangle $\operatorname{Conv}(1, \omega, \omega^2)$.

Example 1.12. Let $N = \mathrm{diag}(\frac{75}{4}i, -7-\frac{21}{4}i, 7-\frac{21}{4}i)$. Note that the origin is the inner center of the triangle consisting of the diagonal entries of N. But $W_2^\Pi(N)$ is not convex. See Figure 3.

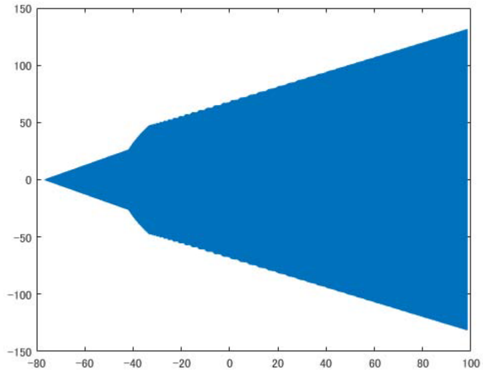

Figure 3: $W_2^\Pi(N)$

Chapter 2

Cyclic Weighted Shift Matrices with Reversible Weights

2.1 Introduction

The numerical range $W(A)$ of an $n \times n$ matrix A is defined as

$$W(A) = \{\xi^* A \xi : \xi \in \mathbb{C}^n, \xi^*\xi = 1\}.$$

This set $W(A)$ was introduced by Toeplitz and Hausdorff proved the convexity of this compact set. Kippenhahn introduced a birational algebraic geometrical method to study this set $W(A)$. He introduce a real ternary homogeneous form

$$F_A(x, y, z) = \det(x\Re(A) + y\Im(A) + zI_n),$$

where $\Re(A) = (A + A^*)/2$, $\Im(A) = (A - A^*)/(2i)$ for the conjugate transpose A^* of A. He showed that the form F_A completely determines the range $W(A)$. Especially he showed that the convex hull of the points $z = x_0 + iy_0$ ($(x_0, y_0) \in \mathbb{R}^2$) for which the line $x_0x + y_0y + 1 = 0$ is a tangent of the real affine curve $F_A(x, y, 1) = 0$ at some point, coincides the range $W(A)$. The real form $F_A(x, y, z)$ satisfies $F_A(0, 0, 1) = 1$ and the every solutions of the equation $F_A(x_1, y_1, z) = 0$ in z is real for every $(x_1, y_1) \in \mathbb{R}^2$. Recently Plaumann and Vinzant [40] proved that a ternary form $F(x, y, z)$ possessing the above property is expressed as

$$F(x, y, z) = \det(xH_1 + yH_2 + zI_n)$$

by using some real Hermitian matrices H_1 and H_2. Their proof is rather elementary. In [32] it is proved that the matrices $H_1 + iH_2$ can be taken as a cyclic weighted shift matrix if the hyperbolic form F is weakly circular invariant. A strict assertion for an arbitrary hyperbolic form

$$F(x, y, z) = \det(xS_1 + yS_2 + zI_n)$$

has been proved by Helton and Vinnikov in [25]. Using the result in [25], Helton and Spitkovsky [24] proved that the numerical range $W(A)$ of an arbitrary $n \times n$ matrix A has some $n \times n$ complex symmetric matrix S satisfying $W(A) = W(S)$. These results arise a new motivation to consider a question: What matrix A is unitarily similar to a complex symmetric matrix. Especially, what cyclic weighted shift matrix is unitarily similar to a symmetric matrix. In addition, complex symmetric matrices or operators have been studied widely in this few years [2, 18]. In [9] Chien et al provide some unitary matrices which uniformly turn Toeplitz matrices into symmetric matrices. We wish to provide another class of matrices satisfying a similar property.

For an $n \times n$ matrix $A = (a_{ij})$ with the entries $a_{12} = w_1, a_{23} = w_2, \ldots, a_{n-1,n} = w_{n-1}$, $a_{n,1} = w_n$, $a_{ij} = 0$ for (i, j) other than $(i, j) = (1, 2), \ldots, (n-1, n), (n, 1)$ is called a *weighted shift matrix*. It is given by

$$\left[\begin{array}{c|ccc} & w_1 & & \\ 0 & & \ddots & \\ & & & w_{n-1} \\ \hline w_n & & 0 & \end{array}\right], \tag{2.1.1}$$

where w_j's called the weights. Various interesting properties are known for weighted shift matrices [43]. As it was shown in [2, Lemma 4], the weighted shift matrix

$$\begin{bmatrix} 0 & 8 & 0 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

is not unitarily similar to a complex symmetric matrix.

The characteristic polynomial of a weighted shift matrix is given by

$$\lambda^n - w_1 w_2 \cdots w_n.$$

Hence if none of weights w_j's vanishes, the weighted shift matrix is similar to a diagonal matrix

$$(w_1 w_2 \ldots w_n)^{1/n} \mathrm{diag}(1, \omega, \omega^2, \ldots, \omega^{n-1}),$$

by an invertible matrix $g \in GL(n : \mathbb{C})$, where $(w_1 w_2 \cdots w_n)^{1/n}$ is one of the n^{th} root of $w_1 w_2 \cdots w_n$ in the field $\mathbb{C}$ and $\omega = \exp(2\pi\sqrt{-1}/n)$. In the case one of the weights w_j's vanishes, the weighted shift matrix S is nilpotent. So various study of weighted shift matrices are usually based on the different methods according to $w_1 w_2 \cdots w_n \neq 0$ or $w_1 w_2 \cdots w_n = 0$. However the method used in this chapter does not need the assumption $w_1 w_2 \cdots w_n \neq 0$, our main interests consists in the case $w_1 w_2 \cdots w_n \neq 0$. A weighted shift matrix satisfying this condition is called *cyclic*. A weight sequence $W = (w_1, w_2, \ldots, w_n)$ is called *reversible* if $w_{n-k+1} = w_k$ for $k = 1, 2, \ldots, n$. We mainly treat the matrix (2.1.1) with reversible weights.

2.2 Main result

The Fourier transform $\tilde{A}$ of an $n \times n$ matrix A is defined as U^*AU, where U is the $n \times n$ unitary matrix defined by

$$U = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & 1 & \cdot & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdot & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdot & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdot & \omega^{3(n-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdot & \omega^{(n-1)^2} \end{bmatrix},$$

where $\omega = \exp(2\pi\sqrt{-1}/n)$. The (k, ℓ)-entry $b_{k\ell}$ of the Fourier transform $B = \tilde{A}$ of an $n \times n$ matrix $A = (a_{pq})$ is given by

$$\tilde{b}_{k\ell} = n\, b_{k\ell} = \sum_{p,q=1}^{n} \omega^{-(k-1)(p-1)} \omega^{(\ell-1)(q-1)} a_{pq}.$$

We present our main theorem.

Theorem 2.1. *Let $A = (a_{pq})$ be an $n \times n$ complex matrix. Then the Fourier transform $B = U^*AU$ of A is a complex symmetric matrix if and only if $a_{1,k+1} = a_{n-k+1,1}$ and $a_{1+k,1+\ell} = a_{n+1-\ell,n+1-k}$ for all $k, \ell = 1, 2, \ldots, n-1$. That is,*

$$A = \left[\begin{array}{c|ccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline a_{1n} & & & \\ \vdots & & \tilde{A} & \\ a_{12} & & & \end{array} \right]$$

where $\tilde{A}$ is an $(n-1) \times (n-1)$ complex matrix which is symmetric with respect to the main skew-diagonal line.

For the 5×5 case, A is of the following form:

$$A=\left[\begin{array}{c|cccc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15}\\ \hline a_{15} & a_{22} & a_{23} & a_{24} & a_{25}\\ a_{14} & a_{32} & a_{33} & a_{34} & a_{24}\\ a_{13} & a_{42} & a_{43} & a_{33} & a_{23}\\ a_{12} & a_{52} & a_{42} & a_{32} & a_{22} \end{array}\right].$$

Proof. Suppose $B=U^*AU$ be a complex symmetric matrix. Note that $A=(a_{pq})$ can be partitioned by the following

$$\sum_{\substack{q-p\equiv 1\\ \text{mod } n}}(a_{pq})+\sum_{\substack{q-p\equiv 2\\ \text{mod } n}}(a_{pq})+\cdots\cdots+\sum_{\substack{q-p\equiv n\\ \text{mod } n}}(a_{pq}).$$

For instance $n=3$, A can be broken up by

$$\begin{bmatrix} 0 & a_{12} & 0\\ 0 & 0 & a_{23}\\ a_{31} & 0 & 0\end{bmatrix}+\begin{bmatrix} 0 & 0 & a_{13}\\ a_{21} & 0 & 0\\ 0 & a_{32} & 0\end{bmatrix}+\begin{bmatrix} a_{11} & 0 & 0\\ 0 & a_{22} & 0\\ 0 & 0 & a_{33}\end{bmatrix}.$$

Fix $m=1,\ldots,n$, let $b_{ij}^{(m)}$ be the (i,j)-entry of (a_{pq}) under the discrete Fourier transform where m satisfies $q-p\equiv m \pmod n$. For a matrix M we denote its i^{th} row and j^{th} column by M_{i*} and M_{*j}, respectively. Therefore,

$$\begin{aligned}
b_{ij}^{(m)} &= \frac{1}{n}(U^*)_{i*}\left[\begin{array}{ccc|ccc} & & & a_{1,m+1} & & \\ & & & & \ddots & \\ & & & & & a_{n-m,n}\\ \hline a_{n-m+1,1} & & & & & \\ & \ddots & & & & \\ & & a_{n,m} & & & \end{array}\right]U_{*j}\\
&= \frac{\omega^{m(j-1)}}{n}(a_{1,m+1}+a_{2,m+2}\omega^{(j-i)}+\cdots+a_{n-m,m}\omega^{(n-m+1)(j-i)}+\\
&\quad a_{n-m+1,1}\omega^{(n-m)(j-i)}+a_{n-m+2,2}\omega^{(n-m-1)(j-i)}\cdots+a_{n-m,n}\omega^{(n-1)(j-i)})\\
&= \left(\frac{\omega^{m(j-1)}}{n}U\begin{bmatrix} a_{1,m+1}\\ a_{2,m+2}\\ \vdots\\ a_{n-m,m}\\ a_{n-m+1,1}\\ a_{n-m+2,2}\\ \vdots\\ a_{n-m,n}\end{bmatrix}\right)_{j-i+1},
\end{aligned}$$

the $(j-i+1)^{\text{th}}$ component of the above vector where

$$(U^*)_{i*} = \left[1, \omega^{-(i-1)}, \omega^{-2(i-1)}, \ldots, \omega^{-(n-1)(i-1)}\right]$$

and

$$U_{*j} = \begin{bmatrix} 1 \\ \omega^{j-1} \\ \vdots \\ \omega^{(n-1)(j-1)} \end{bmatrix}.$$

Similarly,

$$\begin{aligned} b_{ji}^{(m)} &= \frac{\omega^{m(j-1)}}{n}(a_{n-m+1,1} + a_{n-m,n}\omega^{(j-i)} + \cdots + a_{2,m+2}\omega^{(n-m+1)(j-i)} + \\ &\quad a_{1,m+1}\omega^{(n-m)(j-i)} + a_{n,m}\omega^{(n-m-1)(j-i)} \cdots + a_{n-m+2,2}\omega^{(n-1)(j-i)}) \\ &= \left(\frac{\omega^{m(j-1)}}{n} U \begin{bmatrix} a_{n-m+1,1} \\ a_{n-m,n} \\ \vdots \\ a_{2,m+2} \\ a_{1,m+1} \\ a_{n,m} \\ \vdots \\ a_{n-m+2,2} \end{bmatrix} \right)_{j-i+1}. \end{aligned}$$

Let A_m be the following column vector and $A_m(j)$ be the j^{th} component of this vector,

$$A_m = U \left(\begin{bmatrix} a_{1,m+1} \\ a_{2,m+2} \\ \vdots \\ a_{n-m,m} \\ a_{n-m+1,1} \\ a_{n-m+2,2} \\ \vdots \\ a_{n-m,n} \end{bmatrix} - \begin{bmatrix} a_{n-m+1,1} \\ a_{n-m,n} \\ \vdots \\ a_{2,m+2} \\ a_{1,m+1} \\ a_{n,m} \\ \vdots \\ a_{n-m+2,2} \end{bmatrix} \right) = U \begin{bmatrix} a_{1,m+1} - a_{n-m+1,1} \\ a_{2,m+2} - a_{n-m,n} \\ \vdots \\ a_{n-m,m} - a_{2,m+2} \\ a_{n-m+1,1} - a_{1,m+1} \\ a_{n-m+2,2} - a_{n,m} \\ \vdots \\ a_{n-m,n} - a_{n-m+2,2} \end{bmatrix}. \tag{2.2.1}$$

Note that if $j-i+1 < 0$ then we can choose $j-i+1$ is k where $k \in \{1, 2, \ldots, n\}$ which satisfies $j - i + 1 \equiv k \pmod{n}$. Apply the above argument, we have

$$b_{ij} - b_{ji} = \sum_{m=1}^{n} \frac{\omega^{m(j-1)}}{n} A_m(j-i+1). \tag{2.2.2}$$

Hence, if $a_{1,k+1} = a_{n-k+1,1}$ and $a_{1+k,1+\ell} = a_{n+1-\ell,n+1-k}$ for all $k, \ell = 1, 2, \ldots, n-1$ then $A_m(j) = 0$ for all $j, m = 1, 2, \ldots, n$. So $b_{ij} - b_{ji} = 0$ and this establishes the " If " part.
On the other hand, if B is a complex symmetric matrix, since ω^{j-1} and n are both nonzero, (2.2.2) becomes

$$0 = \sum_{m=1}^{n} \frac{\omega^{(m-1)(j-1)}}{\sqrt{n}} A_m(j - i + 1). \tag{2.2.3}$$

We fix $k \in \{1, 2, \ldots, n\}$ with $j - i + 1 \equiv k \pmod{n}$ for all $i, j = 1, 2, \ldots, n$. Since j varies from 1 to n and (2.2.3), we have

$$U \begin{bmatrix} A_1(k) \\ A_2(k) \\ \vdots \\ A_n(k) \end{bmatrix}$$

is a zero vector. This implies $A_m(k) = 0$ for all $k, m = 1, 2 \ldots, n$ as U is invertible. Again, use the invertibility of U in (2.2.1), we have

$$\begin{bmatrix} a_{1,m+1} \\ a_{2,m+2} \\ \vdots \\ a_{n-m,m} \\ a_{n-m+1,1} \\ a_{n-m+2,2} \\ \vdots \\ a_{n-m,n} \end{bmatrix} - \begin{bmatrix} a_{n-m+1,1} \\ a_{n-m,n} \\ \vdots \\ a_{2,m+2} \\ a_{1,m+1} \\ a_{n,m} \\ \vdots \\ a_{n-m+2,2} \end{bmatrix}$$

is a zero vector for all $k, m = 1, 2 \ldots, n$. So $a_{1,k+1} = a_{n-k+1,1}$ and $a_{1+k,1+\ell} = a_{n+1-\ell,n+1-k}$ for all $k, \ell = 1, 2, \ldots, n-1$. This establishes the " Only if " part and completes the proof. □

The following result can be deduced easily by Theorem 2.1.

Corollary 2.2. *A weighted shift matrix with reversible weights is unitarily similar to a complex symmetric matrix.*

We provide an example of the matrix $A = (a_{pq})$ satisfying Theorem 2.1 where m satisfies $q - p \equiv m \pmod{n}$.

Example 2.3. $n = 6$, $m = 2$.

$$A = \begin{bmatrix} 0 & 0 & w_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & w_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & w_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & w_2 \\ w_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & w_4 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The authors wonder if weighted shift matrices are essentially determined by its ternary form $F_W(x, y, z)$. Such a hypothesis is related with the inverse problem of the construction of a matrix W from the $F_W(x, y, z)$. The formula obtained by Helton and Vinnikov in [25, 41] provide a strong tool to treat this subject. The following result would be the first step of our study in this line.

Corollary 2.4. *Let W be an $n \times n$ weighted cyclic shift matrix with reversible weight $\omega_1, \omega_2, \ldots, \omega_2, \omega_1$ and n is odd. Suppose that the curve $F_W(x, y, z) = 0$ has no singular points and $\Im(W)$ has n distinct non-zero eigenvalues $\beta_1, \beta_2, \ldots, \beta_n$. Then there exists a real symmetric matrix S_1 satisfying*

$$\det(xS_1 + y\,\mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_n) + zI_n) = F_W(x, y, z)$$

where S_1 is provided by the Helton-Vinnikov Theorem and $S_1 + i\mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_n)$ is unitarily similar to W.

Proof. By Theorem 2.1, the matrix W is unitarily similar to a complex symmetric matrix. Under this condition and the assumption that the curve $F_W(x, y, z) = 0$ has no singular points, Theorem 7 of [41] guarantees that there is one pair of real symmetric matrix S_1 and S_2 satisfying

$$\det(xS_1 + yS_2 + zI_n) = \det(x\Re(W) + y\Im(W) + zI_n)$$

and $S_1 + iS_2$ is unitarily similar to W. To apply their theorem we assume one standard condition $\Im(W)$ has n distinct non-zero roots. □

Remark 2.5. *The condition "n is odd" in the above corollary is crucial. In the case n is even, the curve $F_W(x, y, z)$ has singular points provided that the weights of W are reversible* [8].

Chapter 3

The Boundary of The q-Numerical Range of Some Toeplitz Nilpotent Matrices

3.1 Introduction

Let A be a bounded linear operator on a complex Hilbert space H. The numerical range of A is defined and denoted by

$$W(A) = \{\langle A\xi, \xi\rangle : \xi \in H, ||\xi|| = 1\}. \tag{3.1.1}$$

In 1919, Hausdorff proved the convexity of this range. The numerical radius $w(A)$ of A is defined as

$$\sup\{|\langle A\xi, \xi\rangle| : \xi \in H, ||\xi|| = 1\}.$$

The various interesting results are known for the radius $w(A)$ and the numerical radius norms on the operator spaces. In this chapter we mainly treat the case H is a finite-dimensional space $\mathbb{C}^n$ of column vectors with the standard inner product $\langle \xi, \eta\rangle = \eta^*\xi$. In this setting, the numerical range satisfies

$$\sigma(A) = \{\lambda \in \mathbb{C} : \det(\lambda I_n - A) = 0\} \subset W(A),$$

$$W(A + \lambda I_n) = \{\lambda + z : z \in W(A)\},$$

$$W(A) = \cap_{0\leq\theta\leq 2\pi}\{z \in \mathbb{C} : \Re(ze^{-i\theta}) \leq \lambda_1(\Re(e^{-i\theta}A))\},$$

where $\Re(B) = (1/2)(B + B^*)$ and $\lambda_1(G)$ is the largest eigenvalue of a Hermitian matrix G.

Goldberg and Strauss [22] introduced the C-numerical range $W_C(A)$ of A as

$$W_C(A) = \{\mathrm{tr}(CU^*AU) : U^*U = I_n\} \tag{3.1.2}$$

where C is an arbitrary $n \times n$ matrix. Cheung and Tsing [6] proved the star-shapedness of $W_C(A)$ with respect to the point $1/n\,\mathrm{tr}(C)\mathrm{tr}(A)$. By using this property Glaser et al [21] developed a numerical algorithm to plot the boundary of $W_C(A)$ and applied it to NMR techniques. If C is a rank one orthogonal projection, the range $W_C(A)$ is reduced to the classical numerical range $W(A)$. In the case A, C are normal matrices, the range $W_C(A)$ is characterized as

$$W_C(A) = \{\sum_{i,j=1}^{n} a_i c_j w_{ij} : (w_{ij}) \in \Omega_n\},$$

where $\sigma(A) = \{a_1, a_2, \ldots, a_n\}$, $\sigma(C) = \{c_1, c_2, \ldots, c_n\}$ and

$$\Omega_n = \{(|u_{ij}|^2 : (u_{ij})\,\text{is an}\,n \times n\,\text{unitary matrix}\}.$$

The above Ω_n is a typical set of entrywise nonnegative matrices. By using the above characterization, the boundary of the range $W_C(A)$ for 3×3 normal matrices are closely analyzed [37]. Tsing [45] proved the convexity of $W_C(A)$ in the case C is a rank one matrix. We consider the 2-dimensional space V containing the ranges $C(\mathbb{C}^n)$, $C^*(\mathbb{C}^n)$. We assume that $||C|| = 1$. Then the operator C restricted to V is unitarily similar to

$$\begin{bmatrix} q & \sqrt{1-|q|^2} \\ 0 & 0 \end{bmatrix}$$

for some $q \in \mathbb{C}$ with $|q| \leq 1$. Using this characterization, the range $W_C(A)$ for a rank-one matrix C is characterized as

$$W_q(A) = \{\eta^* A\xi : \xi, \eta \in \mathbb{C}^n, \xi^*\xi = \eta^*\eta = 1, \eta^*\xi = q\}. \tag{1.3}$$

This range satisfies $W_{cq}(A) = cW_q(A)$ for any $|c| = 1$. So we usually assume that $0 \leq q \leq 1$. If $q = 1$, the range $W_1(A)$ is reduced to $W(A)$. For $0 \leq q < 1$, the range $W_q(A)$ satisfies

$$q\sigma(A) \subset W_q(A), \quad W_q(A + \lambda I_n) = \{q\lambda + z : z \in W_q(A)\}.$$

Boundary points of the range $W(A)$ of an $n \times n$ matrix A lie on an algebraic curve of degree $\leq n(n-1)$ or its bitangents. Boundary points of $W_q(A)$ also lie on an algebraic curve. But its degree is supposed to be so high.

In [?] Duan points out that the notion of numerical range and many of its variants such as local numerical range and q-numerical range play crucial role in characterizing the perfect distinguishability of quantum operations. Such applications bear new motivation to study the q-numerical range. For some more properties of the q-numerical range, we refer [7, 8]. We would expect some relation between the q-numerical ranges and the formula $[n]_q = 1+q+q^2+q^3+\ldots+q^{n-1} = (1-q^n)/(1-q)$ for $-1 < q < 1$. Some relations with the numerical radii and the series $[n]_q$ are known [46]. However no direct relation is known for $W_q(A)$ and $[n]_q$.

We also remark that $W_q(A)$ is a compact convex set of $\mathbb{C} \cong \mathbb{R}^2$. The convexity of this set is useful to analyze this range. The boundary of the unit ball of the 2-dimensional real vector space $\mathbb{R}^2$ with the ℓ^p-norm

$$||\{x_1, x_2\}||_p = (|x_1|^p + |x_2|^p)^{1/p}$$

for $1 < p < \infty$ lies on an algebraic curve if and only if p is a rational number. So the boundary curve is transcendental if p is irrational. Some techniques used there would be useful to study the range $W_q(A)$. In 1984, Tsing [45] provided the following formula

$$W_q(A) = \{q\xi^* A\xi + \sqrt{1-q^2}w\sqrt{\xi^* A^* A\xi - |\xi^* A\xi|^2} : \\ w \in \mathbb{C}, |w| \leq 1 \xi \in \mathbb{C}^n, \xi^*\xi = 1\}. \quad (3.1.3)$$

The function

$$\phi(z) = \max\{\sqrt{\xi^* A^* A\xi - |\xi^* A\xi|^2} : \xi \in \mathbb{C}^n, \xi^*\xi = 1, \xi^* A\xi = z\}$$

($z \in W(A)$) is concave on $W(A)$. By using these properties, Tsing proved the convexity of $W_q(A)$. Based on Tsing's formula, C. K. Li [33] provides a Matlab program to plot $W_q(A)$ numerically. In Section 3, we provide a Mathematica program to plot $W_q(A)$. Its algorithm is basically same with Li's program. A performable algorithm to generate the polynomial $g(x, y)$ for which

$$\{(x, y) \in \mathbb{R}^2 : x + iy \in \partial W_q(A)\} \subset \{(x, y) \in \mathbb{R}^2 : g(x, y) = 0\},$$

$$\{(x, y) \in \mathbb{R}^2 : g(x, y) = 0\} \subset \{(x, y) \in \mathbb{R}^2 : x + iy \in W_q(A)\},$$

$$W_q(A) = \text{Conv}(\{x + iy : (x, y) \in \mathbb{R}^2, g(x, y) = 0\})$$

is given in [7]. We introduce a compact convex sets $\Gamma_0(A)$, $\Gamma(A)$ by

$$\Gamma_0(A) = \{(x_1, x_2, u) \in \mathbb{R}^3 : x_1 + ix_2 \in W(A), u^2 \leq \phi(x_1 + ix_2)^2\},$$

and

$$\Gamma(A) = \{(x_1, x_2, u_1, u_2) \in \mathbb{R}^4 : x_1 + ix_2 \in W(A), u_1^2 + u_2^2 \leq \phi(x_1 + ix_2)^2\}.$$

In a generic case the boundaries of Γ_0, Γ are algebraic hypersurfaces of degree $N = 2n(n-1)^2$. Define an orthogonal projection Π_q of $\mathbb{R}^4$ onto $\mathbb{C} \cong \mathbb{R}^2$ by

$$\Pi_q(x_1, x_2, u_1, u_2) = (qx_1 + \sqrt{1-q^2}u_1) + i(qx_2 + \sqrt{1-q^2}u_2). \tag{3.1.4}$$

Then Tsing's formula is rewritten as

$$W_q(A) = \Pi_q(\Gamma(A)).$$

A general theory of algebraic varieties tell us that the degree of the boundary of $W_q(A)$ is $\leq N(N-1)^2$ (cf. [8]). This upper bound is not sharp for $n = 3$. The above formula provides a principle to compute the equation $g(x, y) = 0$ of the boundary $W_q(A)$. The q-numerical range of some typical 3×3 matrices are given in [8]. Numerical experiments suggest us that the degree of the boundary equation $g(x, y)$ for a generic 3×3 unitarily irreducible matrix is 24. It is rather hard to compute the polynomial $g(x, y)$ for a generic unitarily irreducible 4×4 matrix A by using a standard personal computer. As a first step to treat a generic 4×4 matrix, we treat the following Toeplitz nilpotent matrix

$$N = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.1.5}$$

3.2 Equation of the boundary

The standard method to generate the function $\phi = \phi_A$ on the numerical range $W(A)$ for an $n \times n$ matrix is given by the formula

$$\phi_A(z) = \sqrt{h(z) - |z|^2},$$

$$h(z) = \max\{s : (z, s) \in W(A, A^*A)\}$$

where

$$W(A, A^*A) = \{(z, s) \in \mathbb{C} \times \mathbb{R} : z = \xi^*A\xi, s = \xi^*A^*A\xi, \xi \in \mathbb{C}^n, \xi^*\xi = 1\}.$$

We shall generate a real polynomial $L_{0,A}(X, Y, Z)$ for which the equation $L_{0,A}(X, Y, Z) = 0$ holds for a generic point $(X + iY, Z)$ of the boundary of

$W(A, A^*A)$. As it is mentioned in [7], the algebraic surface $L_{0,A}(X, Y, Z) = 0$ is characterized as the dual surface of the algebraic surface $G_A(x, y, z, 1) = 0$ defined by

$$G_A(x, y, z, t) = \det(x\Re(A) + y\Im(A) + zA^*A + tI_n),$$

where $\Re(A) = (A + A^*)/2$, $\Im(A) = (A - A^*)/(2i)$. By using Sylvester's resultant, we can compute the polynomials G_N and $L_{0,N}$ for the Toeplitz matrix N defined by 3.1.5.

Theorem 3.1. *Suppose that N is the 4×4 Toeplitz matrix given by* (3.1.5). *Then the polynomials G_N and $L_{0,N}$ are given by the following:*

$$\begin{aligned} 4G_N(x, y, z, 1) &= x^2y^2 + y^4 - x^2z^2 - y^2z^2 - 4x^2z - 8y^2z \\ &\quad +4z^3 - 4x^2 - 4y^2 + 16z^2 + 16z + 4, \\ L_{0,N}(X, Y, Z) &= 256(20X^{12} + \ldots + 20X^2Y^{10} + \ldots \\ &\quad +116X^2Y^8Z^2 + \ldots + 4Z^{12}) + \ldots - 52X^2Z - 16Y^2Z \\ &\quad +X^2 + Y^2 + 12Z^2 - Z. \end{aligned}$$

The above degree 12 *polynomial $L_{0,N}(X, Y, Z)$ has* 135 *terms.*

Proof. By direct computations (by using some computer software), we can obtain the explicit expression of the polynomial $G_N(x, y, z, 1)$. For every non-zero vector $(x_0, y_0, z_0) \in \mathbb{R}^3$, we consider the support plane $\Pi(x_0, y_0, z_0)$ of the convex set $W(N, N^*N) \subset \mathbb{C} \times \mathbb{R}^2 \cong \mathbb{R}^3$ defined by

$$\Pi(x_0, y_0, z_0) = \{(x, y, z) \in \mathbb{R}^3, x_0x + y_0y + z_0z = M(x_0, y_0, z_0\},$$

$$M(x_0, y_0, z_0) = \max\{x_0x + y_0y + z_0z : (x + iy, z) \in W(N, N^*N).$$

The value $M(x_0, y_0, z_0)$ is the maximum of the eigenvalues of the Hermitian matrix $x_0\Re(N) + y_0\Im(N) + z_0N^*N$. Thus the boundary of $W(N, N^*N)$ is obtained as the convex hull of the dual surface of the algebraic surface $G_N(x, y, z, 1) = 0$. This polynomial satisfies

$$G_N(-x_0, -y_0, -z_0, M(x_0, y_0, z_0)) = 0.$$

The defining polynomial $L_{0,N}(X, Y, Z) = 0$ of the dual surface of the algebraic surface $G_N(x, y, z, 1) = 0$ is obtained by the elimination of the indeterminates x, y from the equations

$$H(X, Y, Z, x, y) = Z^3G_N(x, y, -\frac{xX}{Z} - \frac{yY}{Z} - \frac{1}{Z}, 1) = 0,$$

$$H_x = \frac{\partial H}{\partial x} = 0, \quad H_y = \frac{\partial H}{\partial y} = 0.$$

We can elimate the indeterminates x, y by successive usage of Sylvester's determinants. We take the simple factor $J(X, Y, Z, y)$ of the resultant of $H(X, Y, Z, x, y)$ and H_x. Then we take the simple factor $L_{0,N}$ of the resultant of $J(X, Y, Z, y)$ and $J_y(X, Y, Z, y)$ with respect to y. In this way we obtain the equation of the dual surface of $G_N(x, y, z, 1) = 0$. To perform this process, Lagrange's interpolation is effective, especially in the second elimination. □

By using the equation $L_{0,A}(X, Y, Z) = 0$ of the boundary of the simultaneous numerical range $W(A, A^*A)$, the equation of the boundary of the convex set $\Gamma(A)$ is given by

$$L_{0,A}(x_1, x_2, x_1^2 + x_2^2 + u_1^2 + u_2^2) = 0.$$

We use the orthogonal projection Π_q of $\mathbb{R}^4$ onto the plane $\mathbb{C} \cong \mathbb{R}^2$ given by (3.1.4). The algorithm to compute the equation of the boundary of $W_q(A)$ is given by the following. We substitute

$$x_1 = \frac{1}{q}(x - \sqrt{1-q^2}u_1), \quad x_2 = \frac{1}{q}(y - \sqrt{1-q^2}u_2)$$

into the polynomial

$$L(x, y, u_1, u_2) = L_{0,A}(x_1, x_2, x_1^2 + x_2^2 + u_1^2 + u_2^2).$$

The polynomial $g(x, y)$ vanishing on the boundary of $W_q(A)$ is obtained by the successive eliminations of u_1, u_2 from the equations

$$M(x, y, u_1, u_2) = L(1/q(x - \sqrt{1-q^2}u_1), 1/q(y - \sqrt{1-q^2}u_2), u_1, u_2),$$

$$M_{u_1}(x, y, u_1, u_2) = 0, M_{u_2}(x, y, u_1, u_2) = 0.$$

We provide the equation of the boundary of $W_q(N)$ for $q = 1599/1601$, $\sqrt{1-q^2} = 80/1601$. This value of q is obtained by a Pythagorean triple $(1599, 80, 1601)$ for which $80/1601$ is rather small.

Theorem 3.2. *Suppose that N is the 4×4 nilpotent matrix given by (1.6) and $q = 1599/1601$. Then every point $x + iy$ of the boundary of $W_q(N)$ ($(x, y) \in \mathbb{R}^2$) satisfies the equation $g(x, y) = 0$ for the following degree 40 polynomial with 253 terms*

$$
\begin{aligned}
g(x,y) &= 2^{26}\cdot 1601^{40}\,(x^2+y^2)^{14}\,(2563201x^2+6400y^2)^2\,(6400x^2+2563201y^2)^4\\
&+2^{25}\cdot 13\cdot 1601^{38}\,(x^2+y^2)^{12}\,(2563201x^2+6400y^2)(6400x^2+2563201y^2)^2\\
&\cdot(330168762708131808512 00x^8 - 1326548367335133886936 9108x^6y^2\\
&-3673981984582525074276 8247x^4y^4 - 4328970570066311850852 4801x^2y^6\\
&-1243585103812678025920 00y^8) + \dots\\
&+2^{14}\cdot 3^{46}\cdot 5^4\cdot 13^{36}\cdot 41^{36}\cdot(811\cdot 2473\cdot 120721\cdot 284689)^2.
\end{aligned}
$$

Proof. The equation $g(x,y)=0$ of the boundary of $W_{1599/1601}(N)$ is obtained by the successive eliminations of u_1, u_2 from the equations $M(x,y,u_1,u_2)=0$ and $M_{u_1}=0$, $M_{u_2}=0$. We take the simple factor $K(x,y,u_2)$ of the resultant of $m(x,y,u_1,u_2)$ and $M_{u_1}(x,y,u_1,u_2)$ with respect to u_1. The total degree of $m(x,y,u_2)$ with respect to x,y is 40. The polynomial $g(x,y)$ is obtained as a simple factor of the resultant of $K(x,y,u_2)$ and $K_{u_2}(x,y,u_2)$ with respect to u_2. These processes essentially coincide with those in [7]. □

By using the above polynomial $g(x,y)$, we shall determine some characteristic invariants of $W_q(N)$ for $q=1599/1601$. We determine the least rectangle R containing $W_{1599/1601}(N)$ with edges parallel to the real and imaginary axes. Since N is a real matrix, the range $W_q(N)$ is symmetric with respect to the real axis. The numerical range $W(A)$ is symmetric with respect to the imaginary axis and the function $\phi(x+iy)$ satisfies $\phi(-x+iy)=\phi(x+iy)$. Hence the range $W_q(N)$ is symmetric with respect to the imaginary axis. So the values

$$
M_x = \max\{\Re(z) : z\in W_q(A)\},\quad M_y=\max\{\Im(z) : z\in W_q(A)\}
$$

are attained respectively on half-lines $\{x : x>0\}, \{iy : y>0\}$. The value M_x for $q=1599/1601$ is the maximum real root of a simple factor

$$
\begin{aligned}
p(x) &= 1726595666980381657907 71204x^8 - 6906382667921526631630 84816x^7\\
&+1035526290327212459458 841624x^6 - 6893449372091030573057 28016x^5\\
&+2141545804290437524684 44805x^4 - 8514557676709384978427 5202x^3\\
&+4270791337192984163838 5601x^2 + 8042994212535089664480 0x\\
&-5359904227656907456320 0
\end{aligned}
$$

of the polynomial $g(x,0)$. The polynomial $p(-x)$ is also a simple factor of $g(x,0)$. The value M_y for $q = 1599/1601$ is the maximum real root of a simple factor

$$q(y) = 1381276533584305326326169632y^8 + 1381276533584305326326169632y^7$$
$$-1383863192750404538040883232y^6 - 1388174291360569890898739232y^5$$
$$+511297035462706296012556812y^4 + 343702202310090276474886408y^3$$
$$-172120143173202689945619204y^2 - 48257965275210537986880 0y$$
$$+42735167843788382245107201$$

of the polynomial $g(0,y)$. The polynomial $q(-y)$ is also a simple factor of $g(0,y)$. The values M_x, M_y are approximately given by

$$M_x \sim 1.0350266, \quad M_y \sim 0.75321029.$$

In Figure 1, we provide a graphic of the curve $g(x,y) = 0$. The outer arc of this figure represents the boundary of $W_{1599/1601}(N)$.

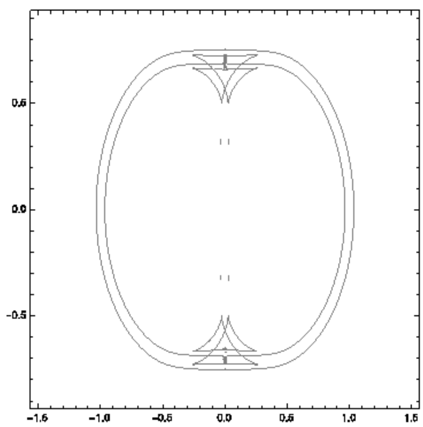

Figure1: $\partial W_q(N)$ and its related envelope curve

3.3 Numerical approximation

We shall provide some codes to plot the q-numerical range of a complex matrix A by using "Mathematica". Our codes depend on Tsing's formula (3.1.3). We expect numerical experiments will be useful for further study. A program to plot the q-numerical range using "Matlab" was provided by [33]. Our program is viewed as its "Mathematica" version. For instance, we treat the matrix (3.1.5).

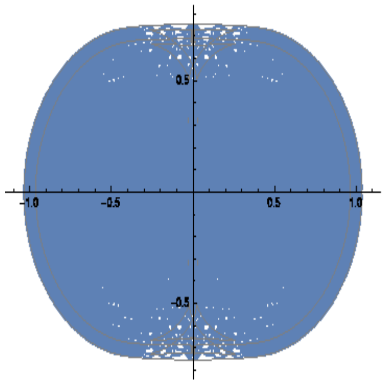

Figure 2

```
A = {{ 0, 1, 0, 1},{0, 0, 1, 0}, {0, 0, 0, 1}, {0, 0, 0, 0}};
A1 = Conjugate[Transpose[A]];
H = (1/2) (A + A1);
G = (-I/2) (A - A1);
K = A1. A;
M1 = 200; M2 = 20; M3 = 20; q = 1599/1601;
  For [ k = 0, k < M1 + 1, k ++, t = k *2 Pi/M1;
        T = Cos[t]*H + Sin[t]*G;
        For[ m = 0, m < M1 + 1, m++, s = m *Pi/(2  M3);
             TT = Cos[s]*T + Sin[s]*K + 5.0 IdentityMatrix[4];
             MM = N[Eigenvectors[TT]][[1]]; M = Abs[MM]^2;
             W = Sqrt[Sum[M[[j]], {j, 1, 4}]];
             v = MM/W;u = Conjugate[v];
             UU = {u};LL = Transpose[{v}];
             p = Re[UU.H. LL]; r = Re[UU. G. LL]; S = Re[UU. K. LL];
             X = p[[1]][[1]]; Y = r[[1]][[1]]; Z = S[[1]][[1]];
             ZZ = Sqrt[1 - q^2]*Sqrt[Z - X^2 - Y^2];
             XX = q * X; YY = q * Y;
             For[ n = 0, n < M2 + 1, n++, s = 2 n * Pi/M2;
                  x[k,m,n] = XX + ZZ*Cos[s] // N;
                  y[k,m,n] = YY + ZZ*Sin[s] // N]]];
Q = Table[{x[k, m, n], y[k, m, n]}, {k, 0, M1}, {m, 0, M3}, {n, 0, M2}];
Q0 = Flatten[Q, 2];
W0 = ListPlot[Q0, PlotRange -> All]
```

In these codes, we may replace M1, M2 and M3 by other numbers. For finer approximations, we need longer computation time. In Figure 2, we merge the graphic produced by these codes and the graphic of the curve in Figure 1. In the above codes, we use the eigenvector of a Hermitian matrices $\cos s(\cos tH + \sin tK) + \sin sK$. In Mathematica's convention, a non-normalized eigenvector corresponding to the eigenvalue of a matrix with the largest modulus is chosen as the first eigenvector. We may meet a case the eigenvalues of the matrix satisfy

$$\lambda_1(\cos s(\cos tH + \sin tK) + \sin sK) \geq 0 > \lambda_n(\cos s(\cos tH + \sin tK) + \sin sK),$$

$$-\lambda_n(\cos s(\cos tH + \sin tK) + \sin sK) \geq \lambda_1(\cos s(\cos tH + \sin tK) + \sin sK) \geq 0,$$

where $\lambda_n(G)$ is the least eigenvalue of a Hermitian matrix G. We can avoid this inconvenience by adding some positive scalar matrix to the matrix

$$\cos s(\cos tH + \sin tK) + \sin sK.$$

In the definition of the vector W, the summation is done for $1 \leq j \leq n$, where the size of the matrix A is $n \times n$, in the above case $n = 4$.

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