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Cauchy Problem for the Euler Equations of a Nonhomogeneous Ideal Incompressible Fluid Ⅲ

ノンホモジニアウスな非圧縮性理想流体のオイラー方程式に対するコーシー問題

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Abstract : It is shown here that the Cauchy problem for the Euler equations of a nonhomogeneous ideal incompressible fluid has a unique solution for a small time interval. In comparison with the previous paper [1] and [2] in references, we discuss the problem under the weaker assumptions to given data, and show the existence of a solution by means of a simple constructive procedure, namely, by proving that a suitable sequence of successive approximations converges.

Key words : Euler equations, nonhomogeneous ideal incompressible fluid

1. Introduction

Let us consider the system of equations :

(1.1) $$\rho_t + v \cdot \nabla \rho = 0 \,,$$

(1.2) $$\rho[v_t + (v \cdot \nabla) v] + \nabla p = \rho f \,,$$

(1.3) $$\text{div}\, v = 0$$

in $Q_T = \mathbb{R}^3 \times [0, T]$, where the density $\rho(x,t)$, the velocity vector $v(x,t) = \big(v_1(x,t), v_2(x,t), v_3(x,t)\big)$ and the pressure $p(x,t)$ are unknowns and $f(x,t)$ is a given vector field of external forces.

In this paper, we solve under the following initial conditions :

(1.4) $$\rho|_{t=0} = \rho_0(x) \,,$$

(1.5) $$v|_{t=0} = v_0(x) \,.$$

Our theorem is the following.

Theorem. *Assume that*

(1.6) $$\rho_0(x) \in C^0(\mathbb{R}^3) \,, \qquad \nabla \rho_0(x) \in H^2(\mathbb{R}^3) \,, \qquad 0 < m \le \rho_0(x) \le M < \infty \,,$$

(1.7) $$\sqrt{\rho_0(x)}\, v_0(x) \in L^2(\mathbb{R}^3) \,, \qquad \nabla v_0(x) \in H^2(\mathbb{R}^3) \,, \qquad \text{div}\, v_0 = 0 \,,$$

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(1.8) $$f(x,t) \in C^0\big([0,T]\,;H^3(\mathbb{R}^3)\big)\ .$$

Then there exists $T^* \in (0,T)$ *such that the problem* $(1.1)-(1.5)$ *has a unique solution* $(\rho, v, p)(x,t)$ *which satisfies*

(1.9) $$\rho(x,t) \in C^0(\mathbb{R}^3 \times [0,T^*])\ ,\ \nabla\rho(x,t) \in C^0\big([0,T^*]\,;H^2(\mathbb{R}^3)\big)\ ,$$

(1.10) $$0 < m \le \rho(x,t) \le M < \infty\ ,$$

(1.11) $$v(x,t) \in C^0\big([0,T^*]\,;H^3(\mathbb{R}^3)\big)\ ,$$

(1.12) $$\nabla p(x,t) \in C^0\big([0,T^*]\,;H^3(\mathbb{R}^3)\big)\ .$$

2. Auxiliary Problems

We assume that $v(x,t) \in C^0\big([0,T]\,;H^3(\mathbb{R}^3)\big)$ is a given function such that div $v = 0$. Hereafter c_j's are the positive constants depending only on the imbedding theorems.

Lemma 2.1. *Under the assumption, problem* (1.1) *with* (1.4) *has a unique solution*

(2.1) $$\rho(x,t) \in C^0(\mathbb{R}^3 \times [0,T])\ , \qquad \nabla\rho(x,t) \in C^0\big([0,T]\,;H^2(\mathbb{R}^3)\big)\ ,$$

which satisfies the estimates

(2.2) $$0 < m \le \rho(x,t) \le M < \infty\ ,$$

and

(2.3) $$\frac{d}{dt}\|\nabla\rho(t)\|_2 \le c_1\|\nabla v(t)\|_2\|\nabla\rho(t)\|_2\ ,$$

where $\|\cdot\|_k = \ \|\cdot\|_{H^k(\mathbb{R}^3)}$. *Moreover, if we put* $\xi(x,t) = \rho(x,t)^{-1}$, *then the estimates*

(2.4) $$M^{-1} \le \xi(x,t) = \rho(x,t)^{-1} \le m^{-1}$$

and

(2.5) $$\frac{d}{dt}\|\nabla\xi(t)\|_2 \le c_1\|\nabla v(t)\|_2\|\nabla\xi(t)\|_2$$

are valid.

Proof. It is well-known that, according to the classical method of characteristics, the solution of problem (1.1) with (1.4) is given by $\rho(x,t) = \rho_0(y(\tau,x,t)|_{\tau=0})$, where $y(\tau,x,t)$ is the solution of the Cauchy problem $\frac{dy}{d\tau} = v(y,\tau)$ with $y|_{\tau=t} = x$. From this, the estimate (2.2) results.

Next, we establish (2.3). Apply the operator $D_x^\alpha = (\partial/\partial x_1)^{\alpha_1}(\partial/\partial x_2)^{\alpha_2}(\partial/\partial x_3)^{\alpha_3}$ on each side of (1.1). Multiplying the result by $D_x^\alpha\rho$, integrating over $\mathbb{R}^3$ and summing over $|\alpha| = \alpha_1+\alpha_2+\alpha_3 = 1,2,3$, we have the equality

$$\frac{1}{2}\frac{d}{dt}\|\nabla\rho\|_2^2 = -\sum_{|\alpha|=1}^{3}\left[\int_{\mathbb{R}^3} v\cdot\nabla(D_x^\alpha\rho)(D_x^\alpha\rho)\,dx + \sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\int_{\mathbb{R}^3} D_x^\beta v\cdot\nabla(D_x^{\alpha-\beta}\rho)(D_x^\alpha\rho)\,dx\right]$$

The first term of the right-hand side is zero, by integration by parts, since div $v = 0$.

The second term can be estimated as follows :

$$\sum_{|\alpha|=1}^{3}\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\left|\int_{\mathbb{R}^3} D_x^{\beta}v\cdot\nabla(D_x^{\alpha-\beta}\rho)(D_x^{\alpha}\rho)\,dx\right| \le c_2\|\nabla v(t)\|_2\|\nabla\rho(t)\|_2^2 \ .$$

Hence, we get estimate (2.3). If we note that $\xi(x,t)$ satisfies the equation $\xi_t+v\cdot\nabla\xi=0$ with $\xi|_{t=0}=\rho_0^{-1}(x)\equiv\xi_0(x)$, the estimates (2.4) and (2.5) directly follows from (2.2) and (2.3). □

Lemma 2.2. *Let $\rho(x,t)$ be the unique solution of (1.1) with (1.4) guaranteed in Lemma 2.1 and $f(x,t)\in C^0\big([0,T]\,;H^3(\mathbb{R}^3)\big)$. Then problem*

$$\operatorname{div}(\xi\nabla p)=\operatorname{div}(f-(v\cdot\nabla)v)=\operatorname{div}f-\sum_{i,j=1}^{3}v_{x_j}^i v_{x_i}^j \tag{2.6}$$

has a unique solution $\nabla p(x,t)\in C^0\big([0,T]\,;H^3(\mathbb{R}^3)\big)$ satisfying

$$\begin{aligned}\|\nabla p(t)\|_3\le c_3[\,&(M+\|\nabla\rho(t)\|_2)(\|\nabla v(t)\|_2^2+\|\nabla f(t)\|_2)\\ &+M(M+\|\nabla\rho(t)\|_2)^3\|\nabla\xi(t)\|_2^3(\|v(t)\|_2^2+\|f(t)\|_0)\,]\ .\end{aligned} \tag{2.7}$$

Proof. We first note that (2.6) comes from applying the divergence operator on both sides of (1.2). It is well-known that (2.6) is solvable in $H^3(\mathbb{R}^3)$. If we multiply (2.6) by p and integrate over $\mathbb{R}^3$, then, by integration by parts, we obtain the equality

$$\int_{\mathbb{R}^3}\xi|\nabla p|^2dx \ = \ \int_{\mathbb{R}^3}(f-(v\cdot\nabla)v)\nabla p\,dx\ .$$

Hence, we get the estimate

$$\|\nabla p(t)\|_0\le M(\|v(t)\|_2^2+\|f(t)\|_0)\ .$$

Noting that (2.6) can be written in the form

$$\Delta p=\rho\left(\operatorname{div}f-\sum_{i,j=1}^{3}v_{x_j}^i v_{x_i}^j\right)-\rho\nabla\xi\cdot\nabla p\ ,$$

and using the inequality $\|u\|_2\le\sqrt{\frac{3}{2}}(\|\Delta u\|_0+\|u\|_0)$, we obtain that for α with $|\alpha|=2$,

$$\|D_x^{\alpha}p\|_2\le\sqrt{\frac{3}{2}}\left(\|D_x^{\alpha}(\rho\operatorname{div}f)\|_0+\left\|D_x^{\alpha}\left(\rho\sum_{i,j=1}^{3}v_{x_j}^i v_{x_i}^j\right)\right\|_0+\|D_x^{\alpha}(\rho\nabla\xi\cdot\nabla p)\|_0+\|D_x^{\alpha}p\|_0\right)\ .$$

Therefore, from $\|\nabla p(t)\|_3\le\sum_{|\alpha|=2}\|D_x^{\alpha}p(t)\|_2+\|\nabla p(t)\|_0$, the interpolation inequality and Young's one, we have the desired estimate. □

Lemma 2.3. *Let $\rho(x,t)$ and $f(x,t)$ be the same as in Lemma 2.2 and $p(x,t)$ the unique solution of (2.6) guaranteed in Lemma 2.2. Then problem*

$$\rho[u_t+(v\cdot\nabla)u]+\nabla p=\rho f\ ,\qquad u|_{t=0}=v_0(x) \tag{2.8}$$

has a unique solution $u(x,t)\in C^0\big([0,T]\,;H^3(\mathbb{R}^3)\big)$. Moreover, $u(x,t)$ satisfies

(2.9) $$\|u(t)\|_0 \le \frac{1}{\sqrt{m}}\left(\|\sqrt{\rho_0}\,v_0\|_0 + \sqrt{M}\int_0^T \|f(t)\|_0\,dt\right) \equiv A_0$$

and

(2.10) $$\frac{d}{dt}\|\nabla u(t)\|_2 \le c_4[\,\|\nabla v(t)\|_2\|\nabla u(t)\|_2 + (m^{-1} + \|\nabla\xi(t)\|_2)\|\nabla p(t)\|_3 + \|\nabla f(t)\|_2\,]\,.$$

Proof. First, multiplying (2.8) by u , integrating over $\mathbb{R}^3$ and noting (2.1), then, by integration by parts, we have the equality

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\rho|u|^2dx = \int_{\mathbb{R}^3}\rho f\cdot u\,dx$$

holds, and thus the inequality (2.9) is valid. Secondly, similary to the proof of Lemma 2.1, we get the equality

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2 = -\sum_{|\alpha|=1}^{3}\left[\int_{\mathbb{R}^3}(v\cdot\nabla D_x^\alpha u)\cdot D_x^\alpha u\,dx + \sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\int_{\mathbb{R}^3}(D_x^\beta v\cdot\nabla D_x^{\alpha-\beta}u)\cdot D_x^\alpha u\,dx\right.$$
$$\left. + \sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\int_{\mathbb{R}^3}(D_x^\beta\xi\, D_x^{\alpha-\beta}\nabla p)\cdot D_x^\alpha u\,dx\right] + \int_{\mathbb{R}^3}D_x^\alpha f\cdot D_x^\alpha u\,dx \equiv \sum_{j=1}^4 I_j\,.$$

Each term is estimated as follows : $I_1 = 0$, $|I_2| \le c_5\|\nabla v(t)\|_2\|\nabla u(t)\|_2^2$,
$|I_3| \le c_6(m^{-1} + \|\nabla\xi(t)\|_2)\|\nabla p(t)\|_3\|\nabla u(t)\|_2$ and $|I_4| \le \|\nabla f(t)\|_2\|\nabla u(t)\|_2$.

Hence the desired estimate is obtained. □

3. Successive Approximations

In order to prove Theorem, we use the method of successive approximations in the following form :

(3.1) $$v^{(0)}(x,t) = 0\,,$$

and for $k = 1,2,3,\cdots$, $\rho^{(k)}(x,t)$, $p^{(k)}(x,t)$, $u^{(k)}(x,t)$ are, respectively, the solution of problems

(3.2) $$\rho_t^{(k)} + v^{(k-1)}\cdot\nabla\rho^{(k)} = 0\,, \qquad \rho^{(k)}|_{t=0} = \rho_0(x)\,,$$

(3.3) $$\mathrm{div}\,(\xi^{(k)}\nabla p^{(k)}) = \mathrm{div}\,(f - (v^{(k-1)}\cdot\nabla)u^{(k-1)})\,,\ \xi^{(k)} = (\rho^{(k)})^{-1}$$

(3.4) $$u_t^{(k)} + (v^{(k-1)}\cdot\nabla)u^{(k)} + \xi^{(k)}\nabla p^{(k)} = f\,, \qquad u^{(k)}|_{t=0} = v_0(x)\,.$$

Finally, let

(3.5) $$v^{(k)} = u^{(k)} - \nabla\psi^{(k)}\,,$$

where $\psi^{(k)}$ is the solution of problem

(3.6) $$\Delta\psi^{(k)} = \mathrm{div}\,u^{(k)}\,.$$

Lemma 3.1. *The sequence* $\{v^{(k)}(x,t)\}_{k=1}^\infty$ *is bounded in* $C^0\big([0,T^*]\,;H^3(\mathbb{R}^3)\big)$ *for a sufficiently small* $T^* \in (0,T)$.

Proof. We note that we can obtain the inequality $\|v^{(k)}(t)\|_3 \le \|u^{(k)}(t)\|_3 + \|\nabla\psi^{(k)}(t)\|_3 \le c_7\|u^{(k)}(t)\|_3$. Let us choose $K \ge \max\{2c_7A_0,\ 4c_7(\|\nabla v_0\|_2 + c_3c_4A_1)\}$, where $A_1 = (m^{-1} + 2\|\nabla\xi_0\|_2)(M + 2\|\nabla\rho_0\|_2)$ $(1 + 8M\|\nabla\xi_0\|_2^3(M + 2\|\nabla\rho_0\|_2)^3)\left(1 + T\|f\|_{C^0([0,T]\,;H^3(\mathbb{R}^3))}\right)$ and define $T^* = \min\{(c_1K)^{-1}\log 2,$ $(c_4K)^{-1}\log 2,\ K^{-2}\}$. Then, from the consequences in the previous section, we find that

$$\sup_{0\le t\le T^*} \|v^{(k)}(t)\|_3 \le K \quad \text{provided that} \quad \sup_{0\le t\le T^*} \|v^{(k-1)}(t)\|_3 \le K .$$

Therefore, by induction, we have the assertion of lemma.

By the direct calculation, we get

Lemma 3.2. *The following estimates hold for* $k = 1, 2, 3, \cdots$.

$$\sup_{0\le t\le T^*} \|\nabla\rho^{(k)}(t)\|_2 \le 2\|\nabla\rho_0\|_2 \equiv A_2\ , \quad \sup_{0\le t\le T^*} \left\|\rho_t^{(k)}(t)\right\|_2 \le KA_2 \equiv A_3\ ,$$

$$\sup_{0\le t\le T^*} \|\nabla\xi^{(k)}(t)\|_2 \le 2\|\nabla\xi_0\|_2 \equiv A_4\ , \quad \sup_{0\le t\le T^*} \left\|\xi_t^{(k)}(t)\right\|_2 \le KA_4 \equiv A_5\ ,$$

$$\sup_{0\le t\le T^*} \|\nabla p^{(k)}(t)\|_3 \le c_3(M + A_2)(1 + 8M(M + A_2)^2A_4^3)\left(K^2 + \|f\|_{C^0([0,T]\,;H^3(\mathbb{R}^3))}\right) \equiv A_6\ ,$$

$$\sup_{0\le t\le T^*} \left\|u_t^{(k)}(t)\right\|_2 \le (m^{-1} + A_4)A_6 + K^2 + \|f\|_{C^0([0,T]\,;H^3(\mathbb{R}^3))} \equiv A_7\ .$$

4. Proof of Theorem

Set $\sigma^{(k)} = \rho^{(k)} - \rho^{(k-1)}$, $\eta^{(k)} = \xi^{(k)} - \xi^{(k-1)}$, $h^{(k)} = u^{(k)} - u^{(k-1)}$, $q^{(k)} = p^{(k)} - p^{(k-1)}$ and $w^{(k)} = v^{(k)} - v^{(k-1)}$. Then we have

$$\sigma_t^{(k)} + v^{(k-1)}\cdot\nabla\sigma^{(k)} = -w^{(k-1)}\cdot\nabla\rho^{(k-1)}, \qquad \sigma^{(k)}|_{t=0} = 0\ , \tag{4.1}$$

$$\eta_t^{(k)} + v^{(k-1)}\cdot\nabla\eta^{(k)} = -w^{(k-1)}\cdot\nabla\xi^{(k-1)}, \qquad \eta^{(k)}|_{t=0} = 0\ , \tag{4.2}$$

$$\begin{aligned} \operatorname{div}(\xi^{(k)}\nabla q^{(k)}) &= -\operatorname{div}(\eta^{(k)}\nabla p^{(k-1)}) - \operatorname{div}\left((w^{(k-1)}\cdot\nabla)v^{(k-1)} + (v^{(k-2)}\cdot\nabla)w^{(k-1)}\right) \\ &= -\operatorname{div}(\eta^{(k)}\nabla p^{(k-1)}) - \sum_{i,j=1}^{3}\left(w_{x_j}^{(k-1),i}v_{x_i}^{(k-1),j} + v_{x_j}^{(k-2),i}w_{x_i}^{(k-1),j}\right) \end{aligned} \tag{4.3}$$

and

$$h_t^{(k)} + (v^{(k-1)}\cdot\nabla)h^{(k)} + \xi^{(k)}\nabla q^{(k)} = -(w^{(k-1)}\cdot\nabla)u^{(k-1)} - \eta^{(k)}\nabla p^{(k-1)}, \quad h^{(k)}|_{t=0} = 0\ . \tag{4.4}$$

In the same way used for getting the estimates of ρ, u and p, we get

$$\|\sigma^{(k)}(t)\|_2 \le A_8\int_0^t \|w^{(k-1)}(s)\|\,ds\ , \qquad \|\eta^{(k)}(t)\|_2 \le A_8\int_0^t \|w^{(k-1)}(s)\|\,ds\ ,$$

where $A_8 = c_8A_2\exp(c_8KT^*)$,

$$\|\nabla q^{(k)}(t)\|_2 \le A_9(\|\eta^{(k)}(t)\|_2 + \|w^{(k-1)}(t)\|_2)\ ,$$

where $A_9 = \max\{A_6A_{10}\ ,\ KA_{10}\}$, $A_{10} = c_9((M + A_2)^2A_4^2 + (M + A_2) + 1)$,

$$\|h^{(k)}(t)\|_2 \le A_{11}\int_0^t (\|\eta^{(k)}(t)\|_2 + \|\nabla q^{(k)}(t)\|_2 + \|w^{(k-1)}(t)\|_2)\,ds\ ,$$

where $A_{11} = c_{10}A_{12}\exp(c_{10}KT^*)$, $A_{12} = \max\{m^{-1} + A_4\ ,\ A_6\ ,\ A_0 + K\}$.

From these inequalities, since $\|w^{(k)}(t)\|_2 \le c_{11}\|h^{(k)}(t)\|_2$, it follows that

$$\|w^{(k)}(t)\|_2 \le A_{13}\int_0^t \|w^{(k-1)}(t)\|_2 ds \le A_{13}^{k-1}\frac{t^{k-1}}{(k-1)!}\sup_{0\le s\le t}\|w^{(1)}(t)\|_2\ ,$$

where $A_{13} = c_{11}(1 + A_8T^*)(1 + A_9)A_{11}$. Consequently,

$$\sup_{0\le t\le T^*}\|w^{(k)}(t)\|_2 \le KA_{13}^{k-1}\frac{t^{k-1}}{(k-1)!}$$

holds. Therefore, we find that

$$\sum_{k=1}^{\infty}\|w^{(k)}\|_{C^0([0,T^*]\,;\,H^2(\mathbb{R}^3))} < \infty\ .$$

This implies that

$$(\rho^{(k)}, p^{(k)}, u^{(k)}, v^{(k)})(x,t) \ \to\ (\rho, p, u, v)(x,t) \quad \text{as} \quad k\to\infty\ ,$$

which satisfies the equations

(4.5) $\qquad \rho_t + v\cdot\nabla\rho = 0\ , \qquad \rho|_{t=0} = \rho_0(x)\ ,$

(4.6) $\qquad \mathrm{div}\Big((v\cdot\nabla)v + \rho^{-1}\nabla p - f\Big) = 0\ ,$

(4.7) $\qquad u_t + (v\cdot\nabla)u + \rho^{-1}\nabla p = f\ , \qquad u|_{t=0} = v_0(x)\ ,$

(4.8) $\qquad \Delta\psi = \mathrm{div}\ u\ ,$

(4.9) $\qquad v = u - \nabla\psi\ .$

Now let us show that $u = v$. Applying the divergence operator on both sides of (4.7) and taking into account (4.6), (4.8) and (4.9) , we get

(4.10) $$(\mathrm{div}\ u)_t + \ v\cdot\nabla(\mathrm{div}\ u) = -\sum_{i,j=1}^{3} v^i_{x_j}\psi_{x_ix_j}\ .$$

Hence, we have the inequality

(4.11) $$\frac{1}{2}\frac{d}{dt}\|\mathrm{div}\ u(t)\|_0^2 \le K\sum_{|\alpha|=2}\|D_x^\alpha\psi(t)\|_0\,\|\mathrm{div}\ u(t)\|_0 \le c_{12}K\|\mathrm{div}\ u(t)\|_0^2\ ,$$

which means div $v = 0$ since div $v_0(x) = 0$.

This completes the proof of Theorem . □

References

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