The existence of special solutions for some unsteady boundary layer problems

ある非定常境界層問題に対する特殊解の存在について

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Abstract : The boundary layer equations for a plane unsteady flow are considered, for which the existence theorems with monotone initial data and with analytic initial data are obtained by Oleinik and Sammartino-Caflish, respectively. Here, without these assumptions, we construct some special smooth solutions.

Key words : boundary layer equation, plane unsteady flow, special solution

1. Introduction

We consider a system of boundary layer equations for a plane unsteady flow of a viscous incompressible fluid

$$\begin{cases} u_t + uu_x + vu_y - \mu u_{yy} = U_t + UU_x, \\ u_x + v_y = 0, \end{cases} \tag{1.1}$$

in the region $D_T = \{(x,y,t)\ ;\ a < x < b,\ 0 < y < \infty,\ 0 < t < T\}$, subject to the conditions

$$\begin{cases} u(x,y,0) = u_0(x,y), \\ u(x,0,t) = v(x,0,t) = 0, \\ u(x,y,t) \to U(x,t) \quad \text{as} \quad y \to \infty. \end{cases} \tag{1.2}$$

As far as we know, there are existence results for (1.1), (1.2) with monotone initial data $(u_0 > 0, u_{0y} > 0)$ in [2] and with analytic initial data in [3]. In this paper, we shall discuss the unique existence of some special smooth solutions to (1.1), (1.2) under the conditions which are not necessarily satisfying these requirements.

We will use the classical notations of the Hölder spaces. Let $I = (0,\infty)$ or $(0,T)$ and $Q_T = (0,\infty) \times (0,T)$. For any non-negative integer *m* and $\alpha \in (0,1)$,

$$C^{m+\alpha}(\bar{I}) = \left\{ \omega(s)\ ;\ \|\omega\|^{(m+\alpha)} = \sum_{j=0}^{m} \left| \left(\frac{\partial}{\partial s}\right)^j \omega \right|^{(0)} + \left| \left(\frac{\partial}{\partial s}\right)^m \omega \right|^{(\alpha)} < +\infty \right\}, \tag{1.3}$$

where

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(1.4) $$|\omega|^{(0)} = \sup_{s\in\bar{I}} |\omega(s)|,$$

(1.5) $$|\omega|^{(\alpha)} = \sup_{s,s'\in\bar{I},s\neq s'} \frac{|\omega(s)-\omega(s')|}{|s-s'|^{\alpha}},$$

and

(1.6) $$C^{m+\alpha}(\overline{Q}_T) = \left\{ w(y,t)\ ;\ \|w\|_T^{(m+\alpha)} \equiv \sum_{2r+s=0}^{m} \left|\left(\frac{\partial}{\partial t}\right)^r \left(\frac{\partial}{\partial y}\right)^s w\right|_T^{(0)} \right.$$
$$\left. + \sum_{2r+s=\max\{0,m-1\}}^{m} \left|\left(\frac{\partial}{\partial t}\right)^r \left(\frac{\partial}{\partial y}\right)^s w\right|_{t,T}^{(\alpha/2)} + \sum_{2r+s=m} \left|\left(\frac{\partial}{\partial t}\right)^r \left(\frac{\partial}{\partial y}\right)^s w\right|_{y,T}^{(\alpha)} < +\infty \right\}$$

where

(1.7) $$|w|_T^{(0)} = \sup_{(y,t)\in\overline{Q}_T} |w(y,t)|,$$

(1.8) $$|w|_{t,T}^{(\alpha/2)} = \sup_{(y,t),(y,t')\in\overline{Q}_T,t\neq t'} \frac{|w(y,t)-w(y,t')|}{|t-t'|^{\alpha/2}},$$

(1.9) $$|w|_{y,T}^{(\alpha)} = \sup_{(y,t),(y',t)\in\overline{Q}_T,y\neq y'} \frac{|w(y,t)-w(y',t)|}{|y-y'|^{\alpha}},$$

(1.10) $$|w|_T^{(\alpha)} = |w|_{t,T}^{(\alpha/2)} + |w|_{y,T}^{(\alpha)}.$$

Our aim is to prove

Theorem 1.1. *Assume that u_0 and U take the form of*

(1.11) $$u_0(x,y) = (Ax+B)\varphi(y)$$

and

(1.12) $$U(x,t) = (Ax+B)\psi(t),$$

respectively, where

(1.13) $$A, B \in \mathbb{R},$$

(1.14) $$\varphi(y) \in C^{2+\alpha}([0,\infty))$$

satisfying

(1.15) $$\varphi(0) = 0, \quad \varphi(y) \to 1 \ \ as \ \ y \to \infty, \quad y\varphi'(y) \in C^{\alpha}([0,\infty)),$$

and

(1.16) $$\psi(t) \in C^{1+\alpha/2}([0,T])$$

satisfying

(1.17) $$\psi(0) = 1.$$

Then there exist $T^* \in (0,T]$ and $h(y,t) \in C^{2+\alpha}(\overline{Q}_{T^*})$ such that

(1.18) $$(u,v)=\left((Ax+B)h(y,t),\ -A\int_0^y h(z,t)dz\right)$$

is a unique solution of (1.1) *in* D_{T^*} *with* (1.2).

We can easily seen that if

(1.19) $$\begin{cases} h_t-\mu h_{yy}=-Ah^2+Ah_y\int_0^y h(z,t)dz+\psi'(t)+A\psi^2(t),\\ h(y,0)=\varphi(y),\\ h(0,t)=0,\\ h(y,t)\to\psi(t)\ \text{ as }\ y\to\infty,\end{cases}$$

has a unique solution in $C^{m+\alpha}(\overline{Q}_{T^*})$ for some $T^*\in(0,T]$, this is certainly a desired function. If we put

(1.20) $$H(y,t)=h(y,t)-\varphi(y)\psi(t),$$

then (1.19) is transformed into

(1.21) $$\begin{cases} H_t-\mu H_{yy}=F(y,t,H)+G(y,t),\\ H(y,0)=0,\\ H(0,t)=0,\\ H(y,t)\to 0\ \ as\ \ y\to\infty,\end{cases}$$

where

(1.22) $$F=A\left[-H^2-2\varphi\psi H+\left(\int_0^y\varphi(z)dz\right)\psi H_y+\varphi'\psi\int_0^y H(z,t)dz+H_y\int_0^y H(z,t)dz\right],$$

(1.23) $$G=A\psi^2+\psi'-A\varphi\psi^2-\varphi\psi'+A\varphi'\left(\int_0^y\varphi(z)dz\right)\psi^2+\mu\varphi''\psi.$$

Therefore, in the following sections, we shall establish the unique solvability of (1.21) in $C^{m+\alpha}(\overline{Q}_{T^*})$ for some $T^*\in(0,T]$.

2. Auxiliary Problem

It is well-known that the solution of

(2.1) $$\begin{cases} w_t-\mu w_{yy}=f(y,t)\in C^{\alpha}(\overline{Q}_T),\\ w(y,0)=0,\\ w(0,t)=0,\\ w(y,t)\to 0\ \text{ as }\ y\to\infty,\end{cases}$$

is given by

(2.2) $$w(y,t)=\int_0^t d\tau\int_{-\infty}^{\infty}\Gamma(y-\eta,t-\tau)g(\eta,\tau)d\eta,$$

where

(2.3) $$\Gamma(y,t)=\frac{1}{\sqrt{4\pi\mu t}}\exp\left(-\frac{y^2}{4\mu t}\right),$$

(2.4) $$g(y,t)=\begin{cases}f(y,t), & y>0,\\ -f(-y,t), & y<0.\end{cases}$$

In what follows, c_j are positive constants independent of T and $\tilde{c}_j$ are positive constants independent of y and t.

Lemma 2.1. *For a solution of* (2.1), *the estimates*

(2.5) $$|w(y,t)|\le \tilde{c}_1 t\exp(-\gamma y^2)\,|f|_T^{(0)},$$

(2.6) $$|w_y(y,t)|\le \tilde{c}_2 t^{(1+\alpha)/2}\exp(-\gamma y^2)\,\|f\|_T^{(\alpha)},$$

(2.7) $$|w_t(y,t)|+|w_{yy}(y,t)|\le \tilde{c}_3 t^{\alpha/2}\exp(-\gamma y^2)\,\|f\|_T^{(\alpha)},$$

(2.8) $$|w_y(y,t)-w_y(y,t')|\le \tilde{c}_4|t-t'|^{(1+\alpha)/2}\exp(-\gamma y^2)\,\|f\|_T^{(\alpha)},$$

(2.9) $$|w_t|_T^{(\alpha)}+|w_{yy}|_T^{(\alpha)}\le c_1\|f\|_T^{(\alpha)}$$

hold, where γ is a positive constant independent of y and t.

Proof. We begin with showing (2.7). By the formula

(2.10) $$w_{yy}(y,t)=\int_0^t d\tau\int_{-\infty}^{\infty}\Gamma_{yy}(y-\eta,t-\tau)\{g(\eta,\tau)-g(y,\tau)\}d\eta$$

and the change of variables $z=\frac{\eta-y}{y\sqrt{4\mu(t-\tau)}}$, we obtain

(2. 11) $$\begin{aligned}&|w_{yy}(y,t)|\\ &\le\frac{2}{\sqrt{\pi}}|f|_{y,T}^{(\alpha)}\int_0^t d\tau\int_{-\infty}^{\infty}\left(\frac{2|y-\eta|^{2+\alpha}}{\{4\mu(t-\tau)\}^{5/2}}+\frac{|y-\eta|^{\alpha}}{\{4\mu(t-\tau)\}^{3/2}}\right)\exp\left(-\frac{(y-\eta)^2}{4\mu(t-\tau)}\right)d\eta\\ &=\frac{2}{\sqrt{\pi}}|f|_{y,T}^{(\alpha)}\int_0^t\{4\mu(t-\tau)\}^{\alpha/2-1}\,d\tau\\ &\quad\times\left\{2y^{3+\alpha}\int_{-\infty}^{\infty}|z|^{2+\alpha}\exp(-y^2z^2)\,dz+y^{1+\alpha}\int_{-\infty}^{\infty}|z|^{\alpha}\exp(-y^2z^2)\,dz\right\}\\ &\le\tilde{c}_5 t^{\alpha/2}|f|_{y,T}^{(\alpha)}\left[y^{3+\alpha}\left\{\int_0^1 z^{2+\alpha}\exp(-y^2z^2)\,dz+\int_1^{\infty}z^{2+\alpha}\exp(-y^2z^2)\,dz\right\}\right.\\ &\quad\left.+y^{1+\alpha}\left\{\int_0^1 z^{\alpha}\exp(-y^2z^2)\,dz+\int_1^{\infty}z^{\alpha}\exp(-y^2z^2)\,dz\right\}\right].\end{aligned}$$

By the way, for any $\beta\ge 0$,

(2. 12) $$\int_0^1 z^{\beta}\exp(-y^2z^2)\,dz\le\int_0^1\exp(-y^2z^2)\,dz=\exp(-\theta^2y^2),\quad 0<\theta<1,$$

and

(2. 13) $$\int_1^{\infty}z^{\beta}\exp(-y^2z^2)\,dz\le\exp\left(-\frac{y^2}{2}\right)\int_1^{\infty}z^{\beta}\exp\left(-\frac{y^2z^2}{2}\right)dz$$

$$= \frac{2^{(1+\beta)/2}}{y^{1+\beta}} \exp\left(-\frac{y^2}{2}\right) \int_{y/\sqrt{2}}^{\infty} \xi^{\beta} \exp(-\xi^2)\, d\xi$$
$$\leq \left\{ 2^{(1+\beta)/2} \int_0^{\infty} \xi^{\beta} \exp(-\xi^2)\, d\xi \right\} y^{-(1+\beta)} \exp\left(-\frac{y^2}{2}\right)$$

Hence, putting $2\gamma = \min\ \{\theta^2, \frac{1}{2}\} > 0$,

$$\text{(2.14)} \qquad |w_{yy}(y,t)| \leq \tilde{c}_6 t^{\alpha/2} |f|_{y,T}^{(\alpha)} \left[y^{3+\alpha} \left\{ \exp(-\theta_1^2 y^2) + y^{-(3+\alpha)} \exp\left(-\frac{y^2}{2}\right) \right\} \right.$$
$$\left. + y^{1+\alpha} \left\{ \exp(-\theta_2^2 y^2) + y^{-(1+\alpha)} \exp\left(-\frac{y^2}{2}\right) \right\} \right]$$
$$\leq \tilde{c}_7 t^{\alpha/2} |f|_{y,T}^{(\alpha)} \{(1 + y^{1+\alpha} + y^{3+\alpha}) \exp(-\gamma y^2)\} \exp(-\gamma y^2)$$
$$\leq \tilde{c}_8 t^{\alpha/2} \exp(-\gamma y^2)\, |f|_{y,T}^{(\alpha)}.$$

If we use the formula

$$\text{(2.15)} \qquad w_t(y,t) = \int_0^t d\tau \int_{-\infty}^{\infty} \Gamma_t\, (y-\eta, t-\tau)\{g(\eta,\tau) - g(\eta,t)\} d\eta,$$

$|w_t(y,t)|$ is estimated similarly.

The estimates (2.5) and (2.6) can be obtained more easily.

We proceed to get (2.8). For the definiteness, we assume that $\tau < t' < t \leq T$.

If $t - t' > t'$,

$$\text{(2.16)} \qquad |w_y(y,t) - w_y(y,t')| \leq |w_y(y,t)| + |w_y(y,t')|$$
$$\leq \tilde{c}_9 \left(t^{(1+\alpha)/2} + t'^{(1+\alpha)/2} \right) \exp\,(-\gamma y^2)\, \|f\|_T^{(\alpha)}$$
$$\leq \tilde{c}_{10} (t-t')^{(1+\alpha)/2} \exp\,(-\gamma y^2)\, \|f\|_T^{(\alpha)}.$$

If $t - t' \leq t'$,

$$\text{(2.17)} \qquad w_y(y,t) - w_y(y,t')$$
$$= \int_{2t'-t}^{t} d\tau \int_{-\infty}^{\infty} \Gamma_y\, (y-\eta, t-\tau)\{g(\eta,\tau) - g(y,\tau)\} d\eta$$
$$- \int_{2t'-t}^{t'} d\tau \int_{-\infty}^{\infty} \Gamma_y\, (y-\eta, t'-\tau)\{g(\eta,\tau) - g(y,\tau)\} d\eta$$
$$+ \int_0^{2t'-t} d\tau \int_{-\infty}^{\infty} \{\Gamma_y(y-\eta, t-\tau) - \Gamma_y(y-\eta, t'-\tau)\}\, \{g(\eta,\tau) - g(y,\tau)\} d\eta$$
$$\equiv I_1 + I_2 + I_3.$$

Since $|I_1|$ and $|I_2|$ are estimated similarly to (2.6), we are just concerned with $|I_3|$.

Set $t'' = t' + \theta(t-t'),\ 0 < \theta < 1$, we have

$$(2.18)\qquad |I_3| = \left|\int_0^{2t'-t} d\tau \int_{-\infty}^{\infty} \left\{\int_{t'}^{t} \Gamma_{ys}(y-\eta, s-\tau)ds\right\}\{g(\eta,\tau)-g(y,\tau)\}d\eta\right|$$
$$\le (t-t')|f|_{y,T}^{(\alpha)} \int_0^{2t'-t} d\tau$$
$$\times \int_{-\infty}^{\infty} \left[\frac{12\mu}{\sqrt{\pi}} \frac{|y-\eta|^{1+\alpha}}{\{4\mu(t''-\tau)\}^{5/2}} + \frac{8\mu}{\sqrt{\pi}} \frac{|y-\eta|^{3+\alpha}}{\{4\mu(t''-\tau)\}^{7/2}}\right] \exp\left\{-\frac{(y-\eta)^2}{4\mu(t''-\tau)}\right\} d\eta$$
$$= \frac{(4\mu)^{(\alpha-1)/2}}{\sqrt{\pi}}(t-t')|f|_{y,T}^{(\alpha)} \int_0^{2t'-t} (t''-\tau)^{(\alpha-3)/2}\, d\tau$$
$$\times \left\{3y^{2+\alpha}\int_{-\infty}^{\infty} |z|^{1+\alpha}\exp(-y^2z^2)\,dz + 2y^{4+\alpha}\int_{-\infty}^{\infty}|z|^{3+\alpha}\exp(-y^2z^2)\,dz\right\}$$
$$\le \tilde{c}_{11}(t-t')^{(1+\alpha)/2}|f|_{y,T}^{(\alpha)}\{(1+y^{2+\alpha}+y^{4+\alpha})\exp(-\gamma y^2)\}\exp(-\gamma y^2)$$
$$\le \tilde{c}_{12}(t-t')^{(1+\alpha)/2}\exp(-\gamma y^2)\,|f|_{y,T}^{(\alpha)}.$$

This completes the proof of lemma, because (2.9) is a standard estimate. □

3. Successive Approximations

We construct the sequence $\{H^{(n)}(y,t)\}$ of successive approximate solutions as follows :

$$(3.1)\qquad H^{(0)}(y,t) = 0,$$

for $n \ge 1$,

$$(3.2)\qquad \begin{cases} H_t^{(n)} - \mu H_{yy}^{(n)} = F(y,t,H^{(n-1)}) + G(y,t), \\ H^{(n)}(y,0) = 0, \\ H^{(n)}(0,t) = 0, \\ H^{(n)}(y,t) \to 0 \quad \text{as} \quad y \to \infty. \end{cases}$$

Lemma 3.1. *If we denote* $|w|_T^{(0)} + |w_y|_T^{(0)} + |w_{yy}|_T^{(0)} + |w_t|_T^{(0)} + |w_y|_{t,T}^{(\alpha/2)} + \|yw_y\|_T^{(\alpha)}$ by $\langle w\rangle_T^{(2,\alpha)}$, *then we have*

$$(3.3)\qquad \|F(y,t,H^{(n-1)})\|_T^{(\alpha)} \le M_1|A|\left(1+\langle H^{(n-1)}\rangle_T^{(2,\alpha)}\right)^2,$$

where M_1 *is a positive function in* $\|\varphi\|^{(1+\alpha)}$, $\|\psi\|^{(\alpha/2)}$ *and* $\|y\varphi'\|^{(\alpha)}$.

Proof. From the form of F, it is sufficient to show the estimate

$$(3.4)\qquad \left\|\frac{1}{y+1}\int_0^y H^{(n-1)}(z,t)dz\right\|_T^{(\alpha)} \le 3\|H^{(n-1)}\|_T^{(\alpha)}.$$

First,

$$(3.5)\qquad \left|\frac{1}{y+1}\int_0^y H^{(n-1)}(z,t)dz\right| \le \frac{y}{y+1}|H^{(n-1)}|_T^{(0)} \le |H^{(n-1)}|_T^{(0)}.$$

Secondly, if $0 < |y-y'| \le 1$,

$$(3.6)\qquad \left|\frac{1}{y+1}\int_0^y H^{(n-1)}(z,t)dz-\frac{1}{y'+1}\int_0^{y'} H^{(n-1)}(z,t)dz\right|$$
$$\leq\left|\frac{1}{y+1}\int_{y'}^{y} H^{(n-1)}(z,t)dz\right|+\left|\frac{y'-y}{(y+1)(y'+1)}\int_0^{y'} H^{(n-1)}(z,t)dz\right|$$
$$\leq\frac{1}{y+1}\left(1+\frac{y'}{y'+1}\right)|y-y'||H^{(n-1)}|_T^{(0)}\leq 2|y-y'|^\alpha|H^{(n-1)}|_T^{(0)},$$

and if $|y-y'|>1$,

$$(3.7)\qquad \left|\frac{1}{y+1}\int_0^y H^{(n-1)}(z,t)dz-\frac{1}{y'+1}\int_0^{y'} H^{(n-1)}(z,t)dz\right|$$
$$\leq 2\left|\frac{1}{y+1}\int_0^y H^{(n-1)}(z,t)dz\right|\leq 2|H^{(n-1)}|_T^{(0)}\leq 2|y-y'|^\alpha|H^{(n-1)}|_T^{(0)}.$$

Finally,

$$(3.8)\qquad \left|\frac{1}{y+1}\int_0^y H^{(n-1)}(z,t)dz-\frac{1}{y+1}\int_0^{y} H^{(n-1)}(z,t')dz\right|$$
$$\leq\left|\frac{1}{y+1}\int_0^y\left\{H^{(n-1)}(z,t)-H^{(n-1)}(z,t')\right\}dz\right|\leq\frac{y}{y+1}|t-t'|^{\alpha/2}|H^{(n-1)}|_{t,T}^{(\alpha/2)}.\qquad\square$$

If we put $K_1(T)=T^{1/2}+T^{\alpha/2}+T^{(1+\alpha)/2}+T,$ we obtain from Lemma 2.1,

$$(3.9)\qquad \langle H^{(n)}\rangle_T^{(2,\alpha)}\leq c_2K_1(T)\left[\|F(y,t,H^{(n-1)})\|_T^{(\alpha)}+\|G(y,t)\|_T^{(\alpha)}\right]$$
$$\leq c_2K_1(T)\left[M_1|A|\left(1+\langle H^{(n-1)}\rangle_T^{(2,\alpha)}\right)^2+M_2\right]$$

and

$$(3.10)\qquad \left|H_t^{(n)}\right|_T^{(\alpha)}+\left|H_{yy}^{(n)}\right|_T^{(\alpha)}\leq c_1\left[M_1|A|\left(1+\langle H^{(n-1)}\rangle_T^{(2,\alpha)}\right)^2+M_2\right],$$

where M_2 is a positive function in $|A|$, $\|\varphi\|^{(2+\alpha)}$, $\|\psi\|^{(1+\alpha/2)}$ and $\|y\varphi'\|^{(\alpha)}$.

Hence there exists $T_0\in(0,T]$ such that

$$(3.11)\qquad \langle H^{(n)}\rangle_{T_0}^{(2,\alpha)}\leq R_1\quad\text{provided that}\quad\langle H^{(n-1)}\rangle_{T_0}^{(2,\alpha)}\leq R_1,$$

and thus

$$(3.12)\qquad \left|H_t^{(n)}\right|_{T_0}^{(\alpha)}+\left|H_{yy}^{(n)}\right|_{T_0}^{(\alpha)}\leq c_1[M_1|A|(1+R_1)^2+M_2]\equiv R_2.$$

Then, by induction, we have

Lemma 3.2. *For* $n\geq 0$, $H^{(n)}(y,t)\in B_{T_0}=\Big\{w\in C^{2+\alpha}\left(\overline{Q}_{T_0}\right)\ ;\ w(0,t)=0\ ,$

$w(y,t)\to 0$ *as* $y\to\infty$, $\langle w\rangle_{T_0}^{(2,\alpha)}\leq R_1$, $|w_t|_{T_0}^{(\alpha)}+|w_{yy}|_{T_0}^{(\alpha)}\leq R_2\Big\}.$

4. Unique Solvability of (1.21)

Let us consider the difference $\tilde{H}^{(n)} = H^{(n)} - H^{(n-1)}$, which satisfies

$$(4.1)\qquad \begin{cases} \tilde{H}_t^{(n)} - \mu \tilde{H}_{yy}^{(n)} = \tilde{F}\big(y, t, \tilde{H}^{(n-1)}\,; H^{(n-1)}, H^{(n-2)}\big), \\ \tilde{H}^{(n)}(y,0) = 0, \\ \tilde{H}^{(n)}(0,t) = 0, \\ \tilde{H}^{(n)}(y,t) \to 0 \ \text{ as } \ y \to \infty, \end{cases}$$

where

$$(4.2)\qquad \tilde{F} = A\Bigg[-(H^{(n-1)} + H^{(n-2)})\tilde{H}^{(n-1)} + 2\varphi\psi\tilde{H}^{(n-1)} + \left(\int_0^y \varphi\, dz\right)\psi\tilde{H}_y^{(n-1)} + \varphi'\psi\int_0^y \tilde{H}^{(n-1)}\, dz + H_y^{(n-1)}\int_0^y \tilde{H}^{(n-1)}\, dz + \tilde{H}_y^{(n-1)}\int_0^y H^{(n-2)}\, dz\Bigg].$$

If we use

$$(4.3)\qquad \|\psi\|^{(\alpha/2)} \le 2T_0^{1-\alpha/2}\|\psi'\|^{(0)} \quad \text{and} \quad \|w\|_{T_0}^{(\alpha)} \le 2\big(T_0 + T_0^{1-\alpha/2}\big)\|w_t\|_{T_0}^{(\alpha)},$$

we derive

$$(4.4)\qquad \|\tilde{F}\|_{T_0}^{(\alpha)} \le c_3 M_3 |A| K_2(T_0)\left(\|\tilde{H}^{(n-1)}\|_{T_0}^{(2+\alpha)} + \left\|y\tilde{H}_y^{(n-1)}\right\|_{T_0}^{(\alpha)}\right),$$

where $M_3 = (\|\varphi\|^{(1+\alpha)} + \|y\varphi'\|^{(\alpha)})\|\psi\|^{(1+\alpha/2)} + R_1 + R_2$ and $K_2(T_0) = T_0 + T_0^{1-\alpha/2}$.

Therefore, from Lemma 2.1, it follows that

$$(4.5)\qquad \begin{aligned} \|\tilde{H}^{(n)}\|_{T_0}^{(2+\alpha)} + \left\|y\tilde{H}_y^{(n)}\right\|_{T_0}^{(\alpha)} &\le c_4\{1 + K_1(T_0)\}\|\tilde{F}\|_{T_0}^{(\alpha)} \\ &\le c_5 M_3 |A|\{1 + K_1(T_0)\}K_2(T_0)\left(\|\tilde{H}^{(n-1)}\|_{T_0}^{(2+\alpha)} + \left\|y\tilde{H}_y^{(n-1)}\right\|_{T_0}^{(\alpha)}\right). \end{aligned}$$

Putting $L(|A|, T_0) = c_5 M_3 |A|\{1 + K_1(T_0)\}K_2(T_0)$, we have by induction,

$$(4.6)\qquad \|\tilde{H}^{(n)}\|_{T_0}^{(2+\alpha)} + \left\|y\tilde{H}_y^{(n)}\right\|_{T_0}^{(\alpha)} \le L(|A|, T_0)^{n-1}\left(\|\tilde{H}^{(1)}\|_{T_0}^{(2+\alpha)} + \left\|y\tilde{H}_y^{(1)}\right\|_{T_0}^{(\alpha)}\right).$$

The property of $L(|A|, T_0)$ implies that for some $T^* \in (0, T_0]$,

$$(4.7)\qquad L(|A|, T^*) < 1.$$

Moreover, since

$$(4.8)\qquad \|\tilde{H}^{(1)}\|_{T^*}^{(2+\alpha)} + \left\|y\tilde{H}_y^{(1)}\right\|_{T^*}^{(\alpha)} = \|H^{(1)}\|_{T^*}^{(2+\alpha)} + \left\|yH_y^{(1)}\right\|_{T^*}^{(\alpha)} \le c_6\{1 + K_1(T^*)\}M_2 < +\infty,$$

we find that

$$\sum_{n=1}^{\infty} L\left(|A|, T^{*}\right)^{n-1}\left(\left\|\tilde{H}^{(1)}\right\|_{T^{*}}^{(2+\alpha)}+\left\|y \tilde{H}_{y}^{(1)}\right\|_{T^{*}}^{(\alpha)}\right)<+\infty . \tag{4.9}$$

That is to say, $\{H^{(n)}(y,t)\}$ converges to an element $H(y,t) \in C^{2+\alpha}(\overline{Q}_{T^*})$ as $n \to \infty$. The uniqueness of such a solution is proved by making use of the estimate analogous to (4.5).

Consequently, the proof of Theorem 1.1 is now completed.

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